Department of Physics

Examination paper for TFY4240 Electromagnetic theory

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Examination time (from-to): 9-13

Permitted examination support material: C

Approved calculator
Rottmann: Matematisk Formelsamling (or an equivalent book of mathematical formulas)

Other information:

This exam consists of three problems, each containing several subproblems. In many cases it is possible to solve later subproblems even if earlier subproblems were not solved. Under normal circumstances, each subproblem (1a-3d) will be given approximately equal weight in the grading, except 1b, which may be given higher (up to double) weight.

The exam is an individual, independent work. During the exam it is not permitted to communicate with others about the exam questions, or distribute drafts for solutions. Such communication is regarded as cheating.

The problems are given in English only. Should you have any language problems related to the exam set, do not hesitate to ask. You can answer in English or Norwegian.

Number of pages (front page included): 11 (7 + 4)

Informasjon om trykking av eksamensoppgave

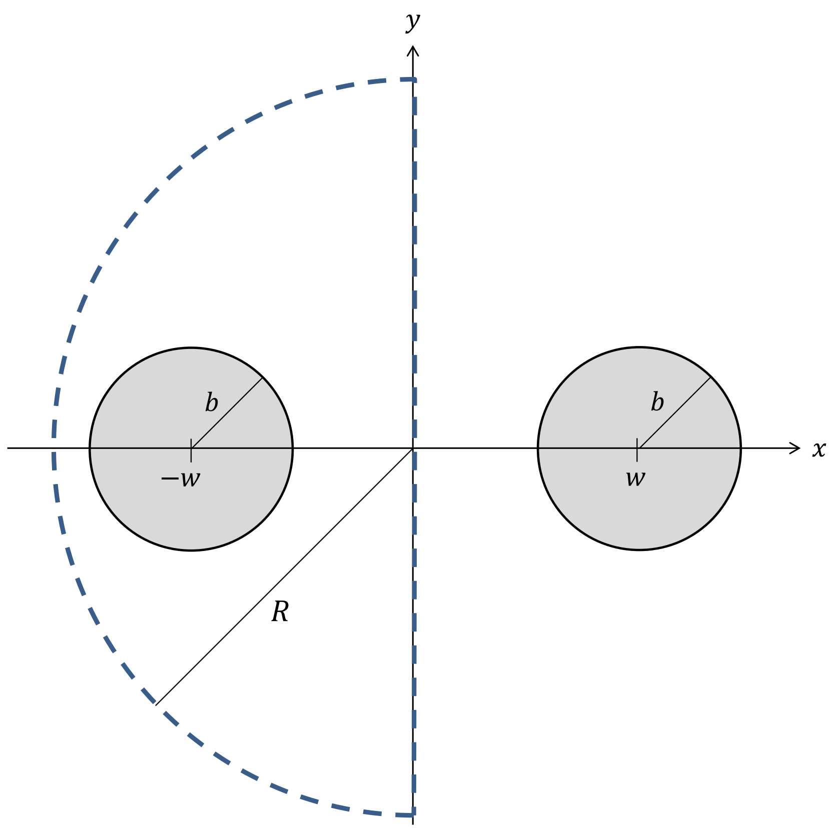
Originalen er:

1-sidig □ **2-sidig X**

sort/hvit X **farger** □

Problem 1.

Consider two fixed, parallel, infinitely long cylinders of radius b whose cross sections are shown as grey disks in the figure below. The cylinder axes are parallel to the z axis (not shown) which points out of the paper plane. Labeling the cylinders 1 and 2, the cylinder axes have x-coordinates $x_1 = -w$, $x_2 = w$, and y-coordinates $y_1 = y_2 = 0$. Cylinder α ($\alpha = 1, 2$) has a surface charge density $\sigma_\alpha(\boldsymbol{r})$ on the curved cylinder surface. The space between the cylinders is vacuum with electric permittivity ϵ_0. The electric field produced by cylinder α at a point $\boldsymbol{r} = (x, y, z)$ will be denoted by $\boldsymbol{E}_\alpha(\boldsymbol{r})$.



a) 1. Consider a finite segment of length L of cylinder 1. Let $\boldsymbol{F}$ be the electric force on this segment due to cylinder 2. Write down an expression for $\boldsymbol{F}$ that takes the form of some kind of spatial integral involving some of the functions introduced above.

2. Assume that the surface charge density σ_α is the same everywhere on the surface. Show that for $\boldsymbol{r}$ outside cylinder α,

$$|\boldsymbol{E}_\alpha(\boldsymbol{r})| = \frac{b|\sigma_\alpha|}{\epsilon_0 s_\alpha} \tag{1}$$

where s_α is the distance from $\boldsymbol{r}$ to the cylinder axis. What is the direction of $\boldsymbol{E}_\alpha(\boldsymbol{r})$? Also determine $\boldsymbol{E}_\alpha(\boldsymbol{r})$ inside the cylinder. (Hint: Make use of symmetry arguments.)

Alternatively, the force $\boldsymbol{F}$ defined above can be expressed as

$$\boldsymbol{F} = \oint_a \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} - \frac{d}{dt}\int_\Omega \frac{\boldsymbol{S}}{c^2}\, d^3r \tag{2}$$

where Ω is an appropriately chosen volume and a is the surface bounding Ω.

b) Again, assume that the surface charge density σ_α is the same everywhere on the surface (with $\sigma_1 \neq \sigma_2$ in general). Then, by symmetry, $\boldsymbol{F}$ must be parallel to $\hat{\boldsymbol{x}}$, so it suffices to consider F_x. Use (2) to calculate F_x. (Hints: Argue that the volume Ω can be chosen as follows: for each z between $-L/2$ and $L/2$ it has the same cross section, taking the shape of a "half-disk" with radius R, as shown by the thick dashed lines in the figure on the previous page. Furthermore, show/argue that in the limit $R \to \infty$ the surface integral in (2) reduces to the contribution from the flat surface at $x = 0$.)

Problem 2.

Consider a simple (i.e. linear and isotropic) non-conducting medium. You may assume that

- the electric permittivity ϵ is real with $\epsilon \geq \epsilon_0$, where ϵ_0 is the vacuum permittivity,
- the magnetic permeability μ is to a very good approximation equal to the vacuum permeability μ_0,
- there is no free charge or free current anywhere.

Then both $\boldsymbol{E}$ and $\boldsymbol{B}$ will satisfy the wave equation in the medium, with wave velocity $v = 1/\sqrt{\epsilon\mu}$, which can also be expressed as $v = c/n$, where $c = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light in vacuum and n is the refractive index, which is real with $n \geq 1$.

a) Consider the plane wave solution (using complex notation, with ω and $\boldsymbol{k}$ real)

$$\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \quad \tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \tag{3}$$

of the wave equations for $\boldsymbol{E}$ and $\boldsymbol{B}$, where $\omega = vk$ (with $k = |\boldsymbol{k}|$).

1. Show that both $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{B}}$ are perpendicular to $\boldsymbol{k}$, and that
$$\tilde{\boldsymbol{B}} = \frac{1}{\omega}(\boldsymbol{k} \times \tilde{\boldsymbol{E}}). \tag{4}$$
2. Evaluate Eq. (4) when $\tilde{\boldsymbol{E}}_0 = \tilde{E}_0\hat{\boldsymbol{y}}$ and $\boldsymbol{k} = k(\cos\theta\,\hat{\boldsymbol{z}} + \sin\theta\,\hat{\boldsymbol{x}})$.

Next, consider two simple non-conducting media 1 and 2. **Each of these media possesses the general properties discussed earlier.** The two media are separated by a flat interface. A plane electromagnetic wave in medium 1 is incident on the interface, giving rise to a reflected wave in medium 1 and a transmitted wave in medium 2.

We choose a coordinate system such that the interface is the xy plane (i.e. at $z = 0$), with medium 1 in the region $z < 0$ and medium 2 in the region $z > 0$. The incident wave has (angular) frequency ω_I and wavevector $\boldsymbol{k}_I$. The corresponding quantities are ω_R and $\boldsymbol{k}_R$ for the reflected wave and ω_T and $\boldsymbol{k}_T$ for the transmitted wave. Then, using the boundary conditions for this system,

$$\begin{aligned} \epsilon_1 E_1^{\perp} &= \epsilon_2 E_2^{\perp}, & (5)\\ B_1^{\perp} &= B_2^{\perp}, & (6)\\ \boldsymbol{E}_1^{\parallel} &= \boldsymbol{E}_2^{\parallel}, & (7)\\ \boldsymbol{B}_1^{\parallel} &= \boldsymbol{B}_2^{\parallel}, & (8) \end{aligned}$$

it can be shown that the exponential factors $e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$ for the three waves must be identical for all times t and all $\boldsymbol{r}$ in the interface, which in turn can be shown to imply the following results:

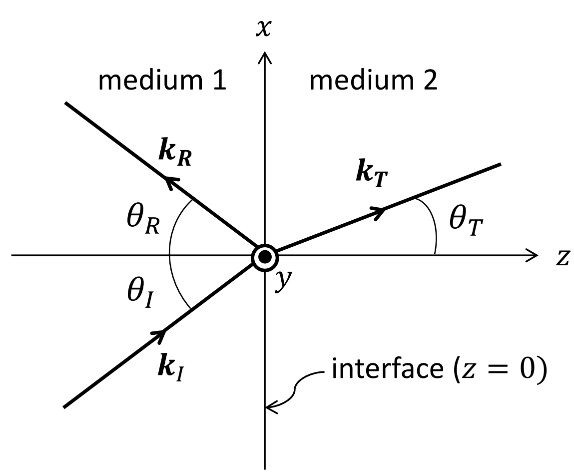
$\omega_T = \omega_R = \omega_I$, (9)

$\boldsymbol{k}_I$, $\boldsymbol{k}_R$ and $\boldsymbol{k}_T$ lie in the same plane (the so-called "plane of incidence"), (10)

$\theta_R = \theta_I$ (the law of reflection), (11)

$n_1 \sin\theta_I = n_2 \sin\theta_T$ (the law of refraction, also known as Snell's law). (12)

The angles θ_I, θ_R, and θ_T here are shown in the figure below, which also shows the directions of the wavevectors (the orientation of the x and y axes in the xy plane have been chosen so that the wavevectors have no y component, and we also made the assumption $n_1 < n_2$ in the figure).



So far, no use has been made of the direction of $\tilde{\boldsymbol{E}}_I$. Now, assume that $\tilde{\boldsymbol{E}}_I$ points along the y axis. Then it can be shown that also $\tilde{\boldsymbol{E}}_R$ and $\tilde{\boldsymbol{E}}_T$ point along the y axis. In other words, the vector $\tilde{\boldsymbol{E}}_0$ in Eq. (3) for the respective waves can be written

$$\tilde{\boldsymbol{E}}_{0I} = \tilde{E}_{0I}\,\hat{\boldsymbol{y}}, \tag{13}$$
$$\tilde{\boldsymbol{E}}_{0R} = \tilde{E}_{0R}\,\hat{\boldsymbol{y}}, \tag{14}$$
$$\tilde{\boldsymbol{E}}_{0T} = \tilde{E}_{0T}\,\hat{\boldsymbol{y}}. \tag{15}$$

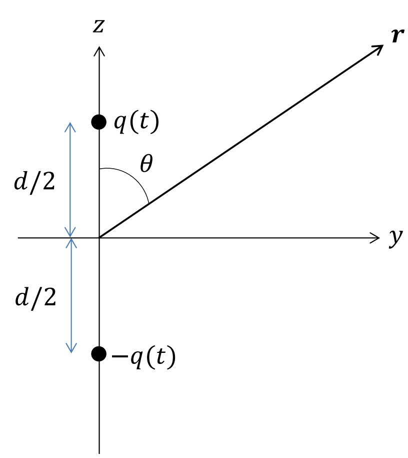
b) Use the boundary conditions (5)-(8) to derive the two equations

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}, \tag{16}$$
$$\tilde{E}_{0I} - \tilde{E}_{0R} = \alpha\beta\,\tilde{E}_{0T}, \tag{17}$$

where $\alpha = \cos\theta_T/\cos\theta_I$ and $\beta = n_2/n_1$. (Remark: It turns out that (6) and (7) lead to the same equation, so only one of these boundary conditions have to be considered.)

c) Assume that $n_1 < n_2$. Determine the phase between the reflected wave and the incident wave as a function of θ_I for $0 \leq \theta_I < \pi/2$, and whether there are any values of θ_I for which the reflected wave vanishes.

Problem 3.



Two tiny metal spheres are separated by a distance d along the z axis (see the figure above). They are connected by an electrically neutral wire, through which a current with angular frequency ω is driven, so that at time t, the charge on the upper sphere is $q(t)$ and the charge on the lower sphere is $-q(t)$, where $q(t) = q_0\cos(\omega t)$. We introduce complex notation by writing $q(t) = \text{Re}\,\tilde{q}(t)$ with $\tilde{q}(t) = q_0\,e^{-i\omega t}$ etc. The current density is given by

$$\tilde{\boldsymbol{j}}(\boldsymbol{r},t) = \hat{\boldsymbol{z}}\frac{d\tilde{q}(t)}{dt}\delta(x)\delta(y)\Theta(d/2 - |z|) \tag{18}$$

where Θ is the Heaviside step function.

a) 1. Treating the metal spheres as points, write down an expression for the charge density $\tilde{\rho}(\boldsymbol{r}, t)$.

2. Make independent calculations of the following two quantities: $\frac{\partial \tilde{\rho}}{\partial t}$ and $\nabla \cdot \tilde{\boldsymbol{j}}$. Give a physical interpretation of the result.

In the following we will be interested in the fields that contribute to radiation, denoted by $\boldsymbol{E}_{\text{rad}}$ and $\boldsymbol{B}_{\text{rad}}$. This means that in your calculations below you may assume that $d \ll r$ and $c/\omega \ll r$, where r is the radial coordinate of the point $\boldsymbol{r} = (r, \theta, \phi)$. (Note that at this stage you should not make any assumptions about the relative magnitude of d and c/ω.) It will be convenient to introduce the wavenumber $k = \omega/c$.

b) We will first consider the potentials $\tilde{V}(\boldsymbol{r}, t)$ and $\tilde{\boldsymbol{A}}(\boldsymbol{r}, t)$. It can be shown (you are not asked to show it) that the vector potential

$$\tilde{\boldsymbol{A}}(\boldsymbol{r}, t) \approx -\frac{icq_0\mu_0}{2\pi} \frac{\sin\left(\frac{kd}{2}\cos\theta\right)}{\cos\theta} \frac{\exp(i(kr - \omega t))}{r} \hat{\boldsymbol{z}}. \tag{19}$$

Find the corresponding expression for the scalar potential $\tilde{V}(\boldsymbol{r}, t)$.

c) It can be shown (you are not asked to show it) that

$$\tilde{\boldsymbol{E}}_{\text{rad}}(\boldsymbol{r}, t) = -\frac{q_0 k}{2\pi\epsilon_0} \tan\theta \sin\left(\frac{kd}{2}\cos\theta\right) \frac{\exp(i(kr - \omega t))}{r} \hat{\boldsymbol{\theta}}. \tag{20}$$

Find $\tilde{\boldsymbol{B}}_{\text{rad}}(\boldsymbol{r}, t)$ (preferably by a calculation not making use of (20)). Compare the expressions for $\tilde{\boldsymbol{E}}_{\text{rad}}$ and $\tilde{\boldsymbol{B}}_{\text{rad}}$; are their similarities and differences as you would have expected?

d) Determine the angular dependence (i.e. the dependence on the angles θ and ϕ) of the resulting radiation pattern, as contained e.g. in the Poynting vector $\boldsymbol{S}$. How does this dependence simplify when $kd \ll 1$? Comment on this simplified dependence.

Formulas

Some formulas that you may or may not need (you should know the meaning of the symbols and possible limitations of validity):

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right) \tag{21}$$

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} \tag{22}$$

$$\frac{d}{du}\Theta(u) = \delta(u) \tag{23}$$

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{\rho(\boldsymbol{r}', t_{\text{ret}})}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{24}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\boldsymbol{j}(\boldsymbol{r}', t_{\text{ret}})}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{25}$$

FUNDAMENTAL CONSTANTS

ϵ_0	$= 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$= 4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	$= 3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
e	$= 1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	$= 9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy : $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum : $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector : $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula : $P = \frac{\mu_0}{6\pi c} q^2 a^2$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$