NTNU **Institutt for fysikk**

Contact during the exam:
Associate Professor Jon Andreas Støvneng replacing Professor Arne Brataas during the exam
Telephone: 45 45 55 33

Exam in TFY4240 Electromagnetic Theory

August 9, 2024
09:00–13:00

Allowed help: Alternativ **C**

A permitted basic calculator and a mathematical formula book (Rottmann or equivalent).

This problem set consists of 6 pages.

This exam consists of 4 problems, each containing several subproblems. There are in total 10 subproblems. Each subproblem (1a-1b-...) will be given equal weight in the grading.

The problems are given in English only. Do not hesitate to ask if you have any language problems related to the exam set. For your answers, you are free to use either English or Norwegian.

Some formulas are given in the appendix on the pages following the last problem.

Good luck!

Problem 1.

a) We consider the system shown in Fig. 1. The metals are grounded, and the potential

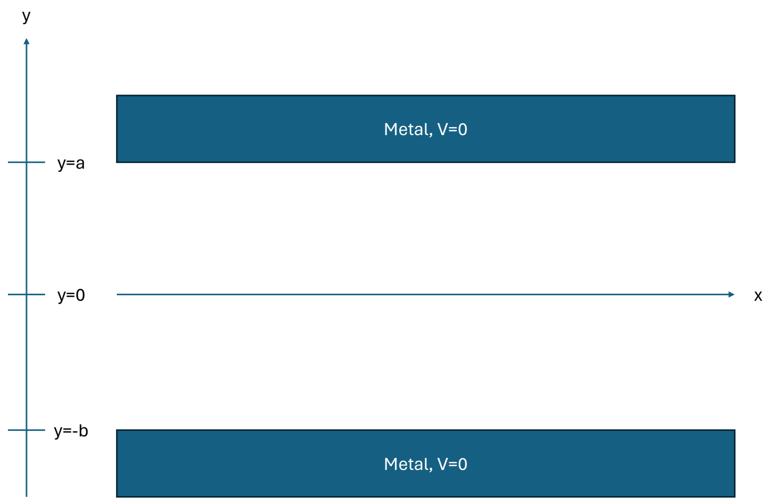

Figure 1: A sheet of surface charge at $y = 0$ is between two grounded metal conductors at $y = a$ and $y = -b$. Although not shown, the system is supposed to be infinitely homogenous in the x and z directions. At $y = 0$, there is a surface charge density $\sigma(x,t)$ that may depend on the spatial coordinate x and the time t..

is $V = 0$ therein. Between the metals, there is a homogenous surface charge density σ at $y = 0$. Above the surface charge density, when $0 < y < a$, the dielectric constant is ϵ_1. Below the surface charge density, when $0 > y > -b$, the dielectric constant is ϵ_2.

We assume the surface charge density is static and varies as $\sigma(x) = \sigma_0 \cos kx$.

Introduce the scalar potential V, solve the equation for V, and determine the electrostatic electric field $\boldsymbol{E} = -\boldsymbol{\nabla} V$ between the metals. Note that the electric field may be inhomogeneous.

b) We consider a material that is linear and isotropic.

The Poynting vector $\boldsymbol{S}$ and the electromagnetic energy density u are determined by

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} \tag{1}$$

$$u = \frac{1}{2}\left(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \boldsymbol{H}\right) . \tag{2}$$

Show that

$$\frac{\partial u}{\partial t} + \boldsymbol{E} \cdot \boldsymbol{J} + \boldsymbol{\nabla} \cdot \boldsymbol{S} = 0 . \tag{3}$$

Explain what this equation means and what the three terms on the left-hand side describe.

Problem 2.

We consider vacuum. In the initial reference frame F1, the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ satisfy the wave equation

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\partial_t^2\right]\boldsymbol{E}(\boldsymbol{r},t)=0\,, \tag{4}$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\partial_t^2\right]\boldsymbol{B}(\boldsymbol{r},t)=0\,. \tag{5}$$

The electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ are also related via Faraday's law (23). We consider another frame of reference F2 with spatial coordinate $\boldsymbol{R}=(X,Y,Z)$ and temporal coordinate τ that is related to the original reference frame with spatial coordinate $\boldsymbol{r}=(x,y,z)$ and temporal coordinate t by the Lorentz transformation

$$\tau=\gamma\left(t-vx/c^2\right)\,, \tag{6}$$

$$X=\gamma\left(x-vt\right)\,, \tag{7}$$

$$Y=y\,, \tag{8}$$

$$Z=z\,, \tag{9}$$

where

$$\gamma=\frac{1}{\sqrt{1-(v/c)^2}} \tag{10}$$

is the Lorentz factor. In other words, the initial reference frame F1 is moving with a velocity v along the x direction with respect to the other reference frame F2. We also have the inverse relationship

$$t=\gamma\left(\tau+vX/c^2\right)\,, \tag{11}$$

$$x=\gamma\left(X+v\tau\right)\,, \tag{12}$$

$$y=Y\,, \tag{13}$$

$$z=Z\,, \tag{14}$$

a) What are the wave equations for the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ in terms of the spatial coordinate $\boldsymbol{R}$ and temporal coordinate τ?

b) Using complex notation, we consider a plane wave

$$\boldsymbol{E}(x,y,z,t)=E_0\hat{\mathbf{y}}e^{i(kx-\omega t)}\,. \tag{15}$$

Compute the frequency Ω a person in reference F2 will observe in terms of the frequency ω in reference F1, the velocity v, and the velocity of light c.

Problem 3.

We consider two materials, 1 and 2, and the interface between them.

a) Derive the boundary conditions (27) and (28).

b) Derive the boundary conditions (29) and (30).

Problem 4.

a) We consider a microwave cavity as shown in Fig. 2. The cavity dimensions are a along the x direction, b along the y direction, and c along the z direction. Metallic plates enclose the cavity.

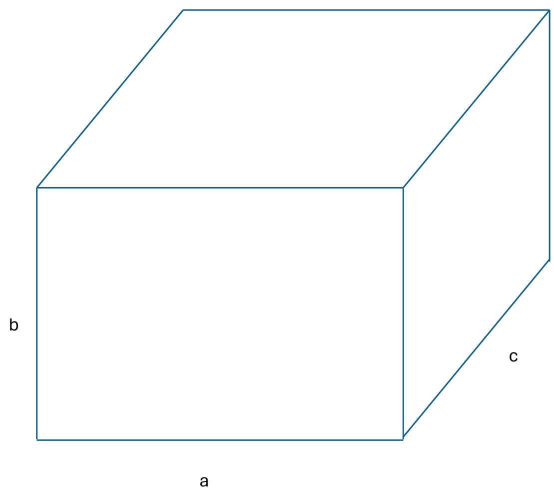

Figure 2: A microwave cavity. Metallic plates enclose the volume abc.

In the Lorentz gauge, and in the absence of free charges, the scalar potential V fulfills the wave equation

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_t^2\right) V(x, y, z) = 0\,. \tag{16}$$

Choose the coordinate system so that one metal plate is located at $x = 0$ and another at $x = a$. Similarly, there are metal plates located at $y = 0$, $y = b$, $z = 0$, and $z = c$. What possible modes for the scalar potential V can exist inside the cavity?

b) We consider a point charge q that is located at position $R = (0, a, 0)$ above a grounded metal plate at $y = 0$ as schematically shown in Fig. 3.

Compute the electric field as a function of position for all locations above the plane.

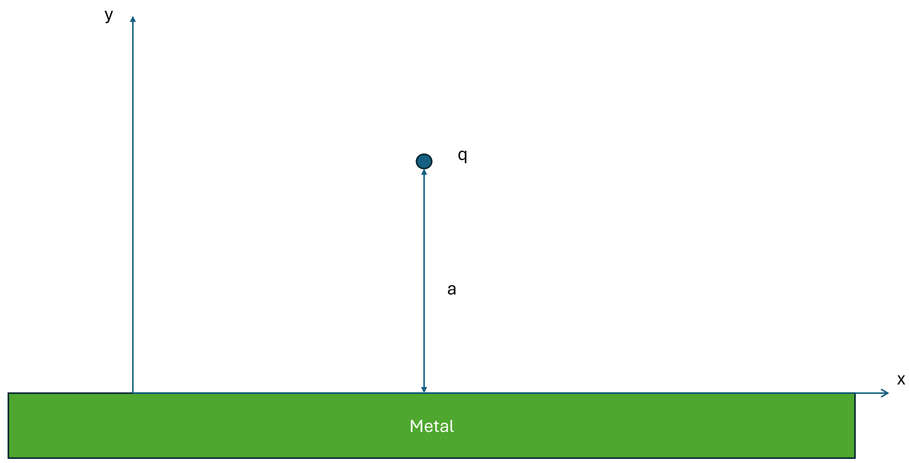

Figure 3: A point charge at $(0, a, 0)$ is above the metal plane at $y = 0$. The metal plane is infinite in the x and z directions. We show here only the projection in the x-y plane.

c) An electron moves with constant velocity v in a circle of radius R so that its position at time t is

$$\boldsymbol{r}_e(t) = R\left(\cos\frac{vt}{R}\hat{\mathbf{x}} + \sin\frac{vt}{R}\hat{\mathbf{y}}\right) , \tag{17}$$

where $\hat{\mathbf{x}}$ is a unit vector along the x direction and $\hat{\mathbf{y}}$ is a unit vector along the y direction.

What is the charge density $\rho(\boldsymbol{r}, t)$ and the charge current density $\boldsymbol{J}(\boldsymbol{r}, t)$? Demonstrate that there is charge conservation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0. \tag{18}$$

d) In the Lorentz gauge, the time-dependent scalar potential V and vector potential $\boldsymbol{A}$ in vacuum are

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0}\int d^3R\frac{\rho(\boldsymbol{R}, t_r)}{|\boldsymbol{R} - \boldsymbol{r}|} , \tag{19}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi}\int d^3R\frac{\boldsymbol{J}(\boldsymbol{R}, t_r)}{|\boldsymbol{R} - \boldsymbol{r}|} . \tag{20}$$

Explain what the retarded time t_r is and why we must use this time t_r in these equations.

A Maxwell‘s Equations

Maxwell‘s equation in vacuum for the electric field $\boldsymbol{E}$, the displacement field $\boldsymbol{D}$, the magnetic induction $\boldsymbol{B}$, and the magnetic field $\boldsymbol{H}$ are

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_f \,, \tag{21}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \,, \tag{22}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \,, \tag{23}$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial D}{\partial t} \,, \tag{24}$$

in terms of the free charge density ρ_f and the free charge current density $\boldsymbol{J}_f$.

B Constitute Relations

In linear and isotropic media, we have

$$\boldsymbol{D} = \epsilon \boldsymbol{E} \,, \tag{25}$$

$$\boldsymbol{B} = \mu H \,, \tag{26}$$

where ϵ is the dielectric constant and μ is the magnetic permeability.

C Boundary Conditions for Electromagnetic Fields

At interfaces between material 1 and material 2, the boundary conditions are

$$\hat{\mathbf{e}}_n \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0 \,, \tag{27}$$

$$\hat{\mathbf{e}}_n \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{K}_s \,, \tag{28}$$

$$\hat{\mathbf{e}}_n \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = \sigma_S \,, \tag{29}$$

$$\hat{\mathbf{e}}_n \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0 \,, \tag{30}$$

where $\hat{\mathbf{e}}_n$ is a unit vector normal to the interface, σ_S is the surface charge density, and $\boldsymbol{K}_s$ is the surface charge current density.

D Spherical Coordinates

In spherical coordinates r, θ, and ϕ, the gradient is

$$\nabla t = \hat{\mathbf{r}} \frac{\partial t}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial t}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{r \sin\theta} \frac{\partial t}{\partial \phi} \,. \tag{31}$$

The divergence is

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\,. \tag{32}$$

The Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,. \tag{33}$$

E Products of matrices

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})\ , \tag{34}$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\,(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}\,(\boldsymbol{A}\cdot\boldsymbol{B})\ . \tag{35}$$

F Integral Theorems

The divergence theorem is

$$\int d^3r\nabla\cdot\boldsymbol{A} = \oint \boldsymbol{A}\cdot d\boldsymbol{S}\,. \tag{36}$$

Stoke‘s theorem (or the curl theorem) is

$$\int (\nabla\times\boldsymbol{A})\cdot d\boldsymbol{S} = \oint \boldsymbol{A}\cdot d\boldsymbol{l}\,. \tag{37}$$

G Some Useful Results

We define $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_1$, $R = |\boldsymbol{R}|$, and $\hat{\mathbf{R}} = \boldsymbol{R}/R$. Then

$$\boldsymbol{\nabla}\cdot\frac{\hat{\mathbf{R}}}{R^2} = 4\pi\delta(\boldsymbol{R})\,, \tag{38}$$

$$\boldsymbol{\nabla}\frac{1}{R} = -\frac{\hat{\mathbf{R}}}{R^2}\,, \tag{39}$$

$$\boldsymbol{\nabla}^2\frac{1}{R} = -4\pi\delta(\boldsymbol{R})\,. \tag{40}$$