Eksamen/Exam 20.12.2000 - Løsningsforslag/suggested solution:

Problem 1

a) The charge conservation $\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$ follows from the two Maxwell's equations $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ and $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$, and the identity $\nabla \cdot (\nabla \times \mathbf{B}) \equiv 0$. We then have: $\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \mathbf{E}) = \mu_0 \left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \right) \equiv 0.$ QED!

With $\mathbf{J} = \sigma\mathbf{E}$, the charge conservation yields $\frac{\partial \rho}{\partial t} = -\sigma \nabla \cdot \mathbf{E} = -\frac{\sigma}{\varepsilon_0}\rho$ and then ρ obeys the simple, first-order, differential equation: $\frac{\partial \ln \rho}{\partial t} = -\frac{\sigma}{\varepsilon_0}$. Direct integration yields the following solution for $t \geq 0$: $\rho(\mathbf{r},t) = \rho_0(\mathbf{r}) \exp\left(-\frac{\sigma}{\varepsilon_0} t \right)$, where $\rho_0(\mathbf{r})$ is the charge distribution at time $t = 0$. We see that any given initial charge distribution very soon decays to zero.

Free charges are not found in the interior of conducting materials because the repulsion of free charges of equal signs causes them to move to the surface of the material. At the same time the attraction of free charges of opposite signs causes them to combine and partly neutralize each other.

b) The fields are given by $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. From $\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$ and $\nabla \times \nabla V \equiv 0$ the following pair of Maxwell’s equations are automatically satisfied: $\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$ and $\nabla \times \mathbf{E} = -\nabla \times \nabla V - \frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\frac{\partial \mathbf{B}}{\partial t}$. QED!

From the last two of Maxwell's equations we obtain:

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \Rightarrow -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = \rho / \varepsilon_0 \Rightarrow$$

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\rho / \varepsilon_0 . \qquad \text{(A)}$$

and: $\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\varepsilon_0\dfrac{\partial\mathbf{E}}{\partial t}\Rightarrow$

$$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}=\mu_0\mathbf{J}-\mu_0\varepsilon_0\left(\nabla\frac{\partial V}{\partial t}+\frac{\partial^2\mathbf{A}}{\partial t^2}\right)\Rightarrow$$

$$\nabla^2\mathbf{A}-\varepsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\mathbf{J}+\nabla\left(\varepsilon_0\mu_0\frac{\partial V}{\partial t}+\nabla\cdot\mathbf{A}\right). \qquad \text{(B)}$$

c) A gauge transform is any change of the potentials V and $\mathbf{A}$ that does *not* change the resulting fields: $\mathbf{E}=-\nabla V-\dfrac{\partial\mathbf{A}}{\partial t}$ and $\mathbf{B}=\nabla\times\mathbf{A}$.

Substituting the Lorentz gauge-condition, $\nabla\cdot\mathbf{A}=-\mu_0\varepsilon_0\dfrac{\partial V}{\partial t}$, into equations (A) and (B) above, we directly obtain the given wave equations:

$$\nabla^2 V-\varepsilon_0\mu_0\frac{\partial^2 V}{\partial t^2}=-\rho/\varepsilon_0 \text{ and } \nabla^2\mathbf{A}-\varepsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\mathbf{J}\text{. QED!}$$

d) The solutions for V and $\mathbf{A}$ are *not* independent because the two wave equations in c) apply only when V and $\mathbf{A}$ are interrelated by the Lorentz gauge-condition. Then ρ and $\mathbf{J}$ in the two wave equations automatically satisfy the charge conservation equation.

Problem 2

a) The potential: $V(\mathbf{r})=\dfrac{1}{4\pi\varepsilon_0}\displaystyle\int\frac{1}{\mathscr{r}}\rho(\mathbf{r}')d\tau'$; where $\mathscr{r}=|\mathbf{r}-\mathbf{r}'|$, is the special solution ("partikulærløsningen") of Poisson's equation: $\nabla^2 V=-\rho/\varepsilon_0$.

If the potential is known, the charge distribution giving rise to the potential is given by: $\rho=-\varepsilon_0\nabla^2 V$. For the given potential we have $V(\mathbf{r})=V(r)$ and directly obtain from the Laplacian in spherical coordinates:

$$\rho(r)=-\varepsilon_0\nabla^2 V(r)=-\frac{\varepsilon_0}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right)=\begin{cases}0\text{; for } r>R\\ \dfrac{Q}{\frac{4}{3}\pi R^3}\text{; for } r\le R.\end{cases}$$

This is a uniform charge distribution of density $\rho=\dfrac{Q}{\frac{4}{3}\pi R^3}$ in a sphere of radius R.

b) With the given formula substituted into the solution given in a) we obtain the multi-pole development:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\text{r}}\rho(\mathbf{r}')d\tau' = \sum_{m=0}^{\infty}\frac{1}{r^{m+1}}\frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{r}')(r')^m P_m(\cos\theta')d\tau' \ .$$

The monopole- and the dipole-terms are, respectively: $V_{mono} = \frac{Q}{4\pi\varepsilon_0 r}$; $Q = \int \rho(\mathbf{r})d\tau$,

and $V_{dip} = \frac{1}{4\pi\varepsilon_0 r^2}\int \rho(\mathbf{r}')r'\cos\theta' d\tau'$.

c) Since $r'\cos\theta' = \mathbf{r}'\cdot\hat{\mathbf{r}}$, we have: $V_{dip}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 r^2}\left(\int \rho(\mathbf{r}')\mathbf{r}'d\tau'\right)\cdot\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$, where

$\mathbf{p} = \int \rho(\mathbf{r})\mathbf{r}d\tau$ is the dipole moment. QED!

With the dipole moment along the *z* axis and spherical coordinates, we have

$V_{dip} = V_{dip}(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$ and obtain the field components from the given formulas for the gradient in spherical coordinates:

$$E_r = -\frac{\partial V_{dip}}{\partial r} = 2\frac{p\cos\theta}{4\pi\varepsilon_0 r^3},\quad E_\theta = -\frac{1}{r}\frac{\partial V_{dip}}{\partial\theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3},\ \text{and}\ E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V_{dip}}{\partial\phi} = 0.$$

d) In coordinate-free form the dipole field is given by:

$$\mathbf{E}_{dip} = -\nabla V_{dip} = -\frac{1}{4\pi\varepsilon_0}\nabla\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right) = -\frac{1}{4\pi\varepsilon_0}\left[\frac{\mathbf{p}}{r^3} - 3\frac{\mathbf{p}\cdot\mathbf{r}}{r^4}\nabla r\right]$$

$$= -\frac{1}{4\pi\varepsilon_0}\left[\frac{\mathbf{p}}{r^3} - 3\frac{(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{r^5}\right] = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right],$$

since $\nabla r = \mathbf{r}/r = \hat{\mathbf{r}}$. QED!

Problem 3

a) Maxwell's equations in integral form follow from the divergence- and the curl-theorems. We here use the general form of the equations (cf. page 10 of the problem set), the corresponding equations in matter follow quite straightforwardly.

$\nabla\cdot\mathbf{B} = 0 \ \Rightarrow\ \oint \mathbf{B}\cdot d\mathbf{a} = 0$ (no magnetic monopoles)

$\nabla\cdot\mathbf{E} = \rho/\varepsilon_0 \Rightarrow \varepsilon_0\oint \mathbf{E}\cdot d\mathbf{a} = Q$ (Gauss' law; $Q = \int \rho d\tau$ is the charge enclosed by the Gaussian surface.)

$\nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}\Rightarrow\oint\mathbf{E}\cdot d\mathbf{l}=-\dfrac{d\Phi}{dt}$ (Faraday's induction law; $\Phi=\int\mathbf{B}\cdot d\mathbf{a}$ is the magnetic flux through the surface enclosed by the loop.)

$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\varepsilon_0\dfrac{\partial\mathbf{E}}{\partial t}\Rightarrow\dfrac{1}{\mu_0}\oint\mathbf{B}\cdot d\mathbf{l}=I_{cond}+I_{displ}$ (Ampère's law; $I_{cond}=\int\mathbf{J}\cdot d\mathbf{a}$ is the conduction current and $I_{displ}=\dfrac{d}{dt}(\varepsilon_0\int\mathbf{E}\cdot d\mathbf{a})$ is the displacement current through the surface enclosed by the Ampèrian loop)

The tangential (in-plane) components of **E** and **H** and the normal components of **B** and **D** are continuous across an interface between two materials without free charges?

b) The field is given by $\mathbf{E}=-\nabla V-\dfrac{\partial\mathbf{A}}{\partial t}$. We only retain terms that do not approach zero faster than $1/r$ for $r\to\infty$. The first term in the expression for V approaches zero as $1/r^2$ and cannot give contributions of the desired form. However, from the second term of V we get a contribution of the desired form when the gradient is taken with respect to the **r** dependence in $\dot{\mathbf{p}}(t-r/c)$. Since $\nabla r=\hat{\mathbf{r}}=\mathbf{r}/r$, we have

$$\mathbf{E}=-\nabla V-\frac{\partial\mathbf{A}}{\partial t}=\frac{\ddot{\mathbf{p}}(t-r/c)\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 cr}\frac{\nabla r}{c}-\frac{\mu_0\ddot{\mathbf{p}}(t-r/c)}{4\pi r}=\frac{\ddot{\mathbf{p}}(t-r/c)\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 c^2 r}\hat{\mathbf{r}}-\frac{\mu_0\ddot{\mathbf{p}}(t-r/c)}{4\pi r}$$
$$=\frac{\mu_0}{4\pi r}\left[(\ddot{\mathbf{p}}(t-r/c)\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\ddot{\mathbf{p}}(t-r/c)\right]=-\frac{\mu_0\ddot{\mathbf{p}}_\perp(t-r/c)}{4\pi r},$$

where $\ddot{\mathbf{p}}_\perp=\ddot{\mathbf{p}}-(\ddot{\mathbf{p}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}$ is the component of $\ddot{\mathbf{p}}$ perpendicular to **r** and we have used that $1/c^2=\mu_0\varepsilon_0$.

c) The given expression follows directly from the result in b) and the vector triple product (see the given formulas): $\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\ddot{\mathbf{p}})=-[\ddot{\mathbf{p}}-(\ddot{\mathbf{p}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]=-\ddot{\mathbf{p}}_\perp$.

For the magnetic field we only get a contribution of the desired form when the curl is taken with respect to the **r** dependence of $\dot{\mathbf{p}}(t-r/c)$, and obtain:

$$\mathbf{B}=\frac{\mu_0}{4\pi r}\nabla\times\dot{\mathbf{p}}(t-r/c)=-\frac{\mu_0}{4\pi rc}\nabla r\times\ddot{\mathbf{p}}(t-r/c)=-\frac{\mu_0}{4\pi rc}\hat{\mathbf{r}}\times\ddot{\mathbf{p}}(t-r/c)=-\frac{\mu_0}{4\pi rc}\hat{\mathbf{r}}\times\ddot{\mathbf{p}}_\perp(t-r/c).$$

Here we have used that: $(\nabla\times\dot{\mathbf{p}})_x=\dfrac{\partial\dot{p}_z}{\partial y}-\dfrac{\partial\dot{p}_y}{\partial z}=-\dfrac{1}{c}\left(\dfrac{\partial r}{\partial y}\ddot{p}_z-\dfrac{\partial r}{\partial z}\ddot{p}_y\right)=-\dfrac{1}{c}(\nabla r\times\ddot{\mathbf{p}})_x$, etc.

d) From the results in b) and c) we see that **E**, **B**, and $\hat{\mathbf{r}}$ form an orthogonal right-handed system in the usual sense. Therefore Poynting's vector is directed radially and given by:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{\mu_0 \ddot{p}_\perp^2(t-r/c)}{(4\pi r)^2 c}\hat{\mathbf{r}} = \frac{\mu_0 \ddot{p}^2(t-r/c)}{(4\pi r)^2 c}\sin^2\theta\ \hat{\mathbf{r}},$$

where we have used that $\ddot{p}_\perp = \ddot{p}\sin\theta$, with θ being the angle between $\ddot{\mathbf{p}}$ and **r**.

Integrating **S** over a sphere of radius r we obtain for the total radiated power:

$$P = \oint \mathbf{S}\cdot d\mathbf{a} = \frac{\mu_0 \ddot{p}^2(t-r/c)}{(4\pi)^2 c}\int_0^\pi \sin^3\theta\,\mathrm{d}\theta \int_0^{2\pi} d\phi = \frac{\mu_0 \ddot{p}^2(t-r/c)}{6\pi c},$$

since $d\mathbf{a} = r^2\sin\theta\, d\theta d\phi\,\hat{\mathbf{r}}$. The θ integral is most easily evaluated by introducing $x = \cos\theta$ as a new integration variable: $\int_0^\pi \sin^3\theta\,\mathrm{d}\theta = \int_{-1}^{1}(1-x^2)\mathrm{d}x = \frac{4}{3}$.
