

TFY 420 ELECTROMAGNETIC THEORY
EXAM 14 DEC 2006 ; SOLUTION

PROBLEM 1

a) We first calculate the dipole moment

It is given that $\vec{p} = \int \vec{r}' \rho(\vec{r}') d\tau'$

For discrete charges this takes the

form $\vec{p} = \sum_i \vec{r}_i q_i = \sum_i (x_i q_i \hat{x} + y_i q_i \hat{y} + z_i q_i \hat{z})$

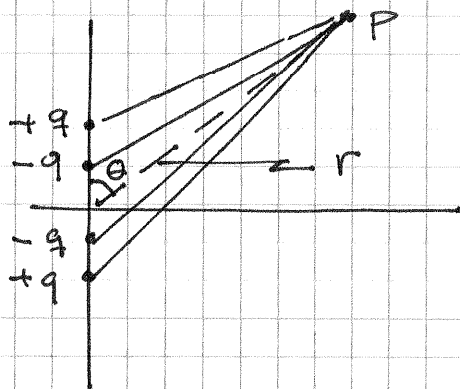
Performing this sum for the given charges gives

$$\begin{aligned}\vec{p} &= \hat{x} (0) + \hat{y} (-2q \cdot a - 2q \cdot (-a)) + \hat{z} (3q \cdot a - qa) \\ &= 2qa \hat{z}\end{aligned}$$

The potential is thus:

$$V(r) = \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2} = \frac{2qa \hat{z} \cdot \hat{r}}{4\pi\epsilon_0 r^2} = \frac{2qa \cos\theta}{4\pi\epsilon_0 r^2}$$

Next; calculate the potential for the four charges along the z-axis.



Here we could simply calculate $V = \sum_1^4 \frac{q_i}{4\pi\epsilon_0 r_i}$

and then expand r_i . I choose to use the expressions given. Thus we calculate

$$Q = \sum_1^4 q_i = +q - q - q + q = 0$$

$$\vec{p} = \sum_1^4 q_i \vec{r}_i = (q(-2a) - q(-a) - q \cdot a + q \cdot 2a) \hat{z} = 0$$

The dipole moment is also zero. The lowest contribution is thus the quadrupole moment

$$\text{We calculate } Q_4 = \sum_i^4 q_i r_i^2 \left(\frac{3}{2} \cos^2 \theta' - \frac{1}{2} \right)$$

θ' is the angle between \vec{r}_i and \vec{r} (see figure 3 in the problem set)

For charges with positive z , $\theta' = \theta$

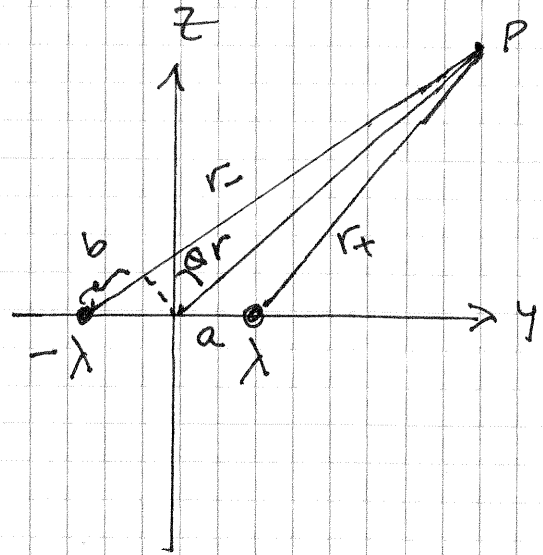
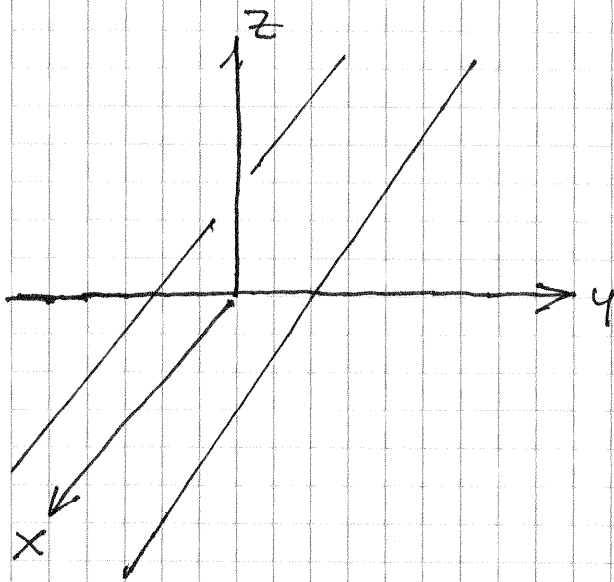
For charges with negative z , $\theta' = \pi - \theta$,

but $\cos^2(\pi - \theta) = \cos^2 \theta$. Thus

$$\begin{aligned} Q_4 &= \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \sum_1^4 r_i^2 q_i \\ &= \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) (q(4a^2 + a^2 + a^2 + 4a^2)) \\ &= (3 \cos^2 \theta - 1) 3qa^2 \end{aligned}$$

and the potential is

$$V(r) = \frac{3qa^2}{4\pi\epsilon_0 r^3} (3 \cos^2 \theta - 1)$$



We can write

$$V(r) = -\frac{\lambda}{2\pi\epsilon_0} \ln r_+ + \frac{\lambda}{2\pi\epsilon_0} \ln r_- + c$$

We can choose $c = 0$, $b = a \sin \theta$

$$\begin{aligned} V(r) &= -\frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_+}{r_-} = -\frac{\lambda}{2\pi\epsilon_0} \ln \frac{r-b}{r+b} \\ &= -\frac{\lambda}{2\pi\epsilon_0} \left(\ln \left(1 - \frac{b}{r}\right) - \ln \left(1 + \frac{b}{r}\right) \right) \end{aligned}$$

We expand the logarithms

$$\begin{aligned} V(r) &= -\frac{\lambda}{2\pi\epsilon_0} \left(-\frac{b}{r} - \frac{b}{r} \right) \\ &= + \frac{\lambda a \sin \theta}{\pi \epsilon_0 r} \end{aligned}$$

OPPGAVE 2

a) Multipliser begge sider av Faradays lov med et flatelement $d\mathbf{A}$ og integrer begge sider over en flate S :

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{A} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

Bruk Stokes' teorem ("curl-teoremet", se vedlegg) på venstre side:

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{A} = \oint \mathbf{E} \cdot d\mathbf{l}$$

På høyre side kan tidsderivasjonen tas utenfor integraltegnet ettersom flaten S holdes fast:

$$- \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} = - \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{A}$$

Dermed er

$$\oint \mathbf{E} \cdot d\mathbf{l} = - \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{A}$$

som vi skulle vise.

Multipliser begge sider av Gauss' lov for magnetfeltet med et volumelement dV og integrer begge sider over et volum:

$$\int (\nabla \cdot \mathbf{B}) dV = 0$$

Bruk divergensteoremet (se vedlegg) på venstre side:

$$\int (\nabla \cdot \mathbf{B}) dV = \oint \mathbf{B} \cdot d\mathbf{A}$$

Dermed er

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0$$

som vi skulle vise.

b) Se f.eks. D. J. Griffiths, *Introduction to electrodynamics* (3rd ed), 7.3.6.

c) Bølgeligningen er egentlig tre ligninger:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \tilde{E}_j = \mu_0 \varepsilon_0 \frac{\partial^2 \tilde{E}_j}{\partial t^2}$$

for $j = x, y, z$. Alle tre gir oss

$$- (k_x^2 + k_y^2 + k_z^2) = - \frac{\omega^2}{c^2}$$

der $c = 1/\sqrt{\mu_0 \varepsilon_0}$ er lyshastigheten i vakuum.

Grensebetingelsen for tangentialkomponenten av \mathbf{E} gir oss de tillatte verdiene av k_x , k_y og k_z .
Inne i de metalliske veggene er $\mathbf{E} = 0$, så vi må ha

$$E_x(y=b) = E_x(z=d) = E_y(x=a) = E_y(z=d) = E_z(x=a) = E_z(y=b) = 0$$

(På de tre flatene $x = 0$, $y = 0$ og $z = 0$ er tangentialkomponenten av \mathbf{E} lik null uavhengig av hva k_x , k_y og k_z er.)

Dermed må både $k_x a$, $k_y b$ og $k_z d$ være lik et helt antall ganger π , dvs

$$k_x = \frac{m\pi}{a} \quad , \quad k_y = \frac{n\pi}{b} \quad , \quad k_z = \frac{l\pi}{d}$$

med heltallige m , n og l . Bare en av disse kan være null om gangen, hvis ikke er $\mathbf{E} = 0$, og vi har ingen elektromagnetisk bølge i det hele tatt.

Tillatte vinkelfrekvenser er med andre ord

$$\omega_{mnl} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{l^2}{d^2}}$$

d) Med $a < d < b$ er de tre laveste modene (011), (110) og (101). Innsatt tallverdier $a = 0.25$ m, $b = 0.40$ m, $d = 0.30$ m og $c = 3 \cdot 10^8$ m/s gir

$$f_{011} = 625 \text{ MHz}$$

$$f_{110} = 708 \text{ MHz}$$

$$f_{101} = 781 \text{ MHz}$$

Problem 3.

a) The time variation of the source is $e^{+i\omega t}$, but at the observation point we have to replace this by $e^{i\omega t_r}$ where t_r is the retarded time $t_r = t - r/c$

$$\text{Thus } e^{i\omega t_r} = e^{i\omega(t - r/c)} = e^{i(\omega t - kr)}$$

$$\text{Since } \omega = kc$$

the conditions are those given in the textbook p 445, 446

1) $d \ll r$; The size of the dipole small compared to the distance r

2) $d \ll \lambda$; λ is the wavelength of the emitted radiation ($d \ll \frac{c}{\omega}$ in the book)

3) $\lambda \ll r$ Wavelength should be small compared to the distance from the dipole to the observation point.

b)

We have to calculate $\vec{B} = \nabla \times \vec{A}$

We use the Levi-Civita notation.

This gives

$$\begin{aligned} B_i &= (\nabla \times \vec{A})_i = \delta_{ijk} \partial_j A_k \\ &= \delta_{ijk} \partial_j \left(i\omega \frac{\mu_0}{4\pi} p_{0k} \frac{e^{i(\omega t - kr)}}{r} \right) \end{aligned}$$

Since we are only interested in the dominating term, we only have to take the derivative of the exponential term. Thus

$$\begin{aligned} B_i &= i\omega \frac{\mu_0}{4\pi} \frac{p_{0k}}{r} \partial_j e^{i(\omega t - kr)} \cdot \delta_{ijk} \\ &= i\omega \frac{\mu_0}{4\pi} \frac{p_{0k}}{r} e^{i(\omega t - kr)} \cdot (-ik) \partial_j r \cdot \delta_{ijk} \end{aligned}$$

$$\partial_j r = \partial_j (\sum x_i^2)^{1/2} = \frac{x_j}{r}$$

$$\begin{aligned} B_i &= \omega k \frac{\mu_0}{4\pi} \frac{e^{i(\omega t - kr)}}{r} \delta_{ijk} x_j \frac{p_{0k}}{r} \\ &= \omega k \frac{\mu_0}{4\pi} \frac{e^{i(\omega t - kr)}}{r} \left[\frac{\vec{r} \times \vec{p}_0}{r} \right]_i \end{aligned}$$

Totally we get

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_0 \omega^2}{4\pi c} (\vec{r} \times \vec{p}_0) \frac{e^{i(\omega t - kr)}}{r}$$

We would get the same result

in the form $-\frac{\mu_0 \rho \omega^2}{4\pi r} e^{i(\omega t - kr)} \frac{1}{r}$ if

we used the ∇ operator in spherical coordinates given in the formula sheet

$$c) \quad \vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B})$$

$$\text{We use } \vec{B} = \frac{1}{c} (\hat{r} \times \vec{E})$$

$$\Rightarrow \vec{S} = \frac{1}{\mu_0 c} (\vec{E} \times (\hat{r} \times \vec{E}))$$

$$= \frac{1}{\mu_0 c} (r^2 E^2 - \vec{E} (\hat{r} \cdot \vec{E}))$$

Since $\vec{E} \perp \hat{r}$ the last term disappears

$$\vec{S} = \frac{1}{\mu_0 c} E^2 \hat{r} = \frac{\sqrt{\epsilon_0 \mu_0}}{\mu_0} E^2 \hat{r}$$

$$= \sqrt{\frac{\epsilon_0}{\mu_0}} E^2 \hat{r}$$

The power radiated through an area \vec{a} is

$$P_a = \vec{S} \cdot \vec{a}$$

The dimensions of $E^2 \cdot a^2 = (\text{Field} \cdot \text{length})^2$

$$= \text{volt}^2$$

and the power P_a is in watts.

Comparing to Ohm's law $P = \frac{V^2}{R}$

We see that $\sqrt{\frac{\mu_0}{\epsilon_0}}$ is a "resistance"

It's value is

$$R_{\text{rad}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi \cdot 10^{-7}}{8,85 \cdot 10^{-12}}} = 377 \Omega$$

Looking at the expression for \vec{B} we see that it can be written as

$$\vec{B} = -\frac{i\omega}{c} (\hat{r} \times \vec{A})$$

$$\text{and from } \vec{E} = -c(\hat{r} \times \vec{B})$$

$$\Rightarrow \vec{E} = i\omega (\hat{r} \times (\hat{r} \times \vec{A}))$$

This gives for the Poynting vector

$$\vec{S} = \frac{1}{\mu_0} (i\omega) \left(-\frac{i\omega}{c}\right) (\hat{r} \times (\hat{r} \times \vec{A})) \times (\hat{r} \times \vec{A})$$

$$\text{Call } \hat{r} \times \vec{A} = \vec{T}$$

$$\vec{S} = \frac{\omega^2}{\mu_0 c} (\hat{r} \times \vec{T}) \times \vec{T} = -\frac{\omega^2}{\mu_0 c} (\vec{T} \times (\hat{r} \times \vec{T}))$$

$$= -\frac{\omega^2}{\mu_0 c} \left(\hat{r} T^2 - \vec{T} (\hat{r} \cdot \vec{T}) \right)$$

||
0

$$\vec{S} = -\frac{1}{\mu_0 c} \omega^2 \hat{r} (\hat{r} \times \vec{A})^2$$

$$= -\sqrt{\frac{\epsilon_0}{\mu_0}} \omega^2 (\hat{r} \times \vec{A})^2 \hat{r}$$

$$\text{Thus } Y = -\sqrt{\frac{\epsilon_0}{\mu_0}} \omega^2$$

We have used complex notation here.
then the Poynting vector should
be written

$$\bar{S} = \frac{1}{\mu_0} (\bar{E} \times \bar{B}^*)$$

and the time averaged Poynting
vector

$$\langle \bar{S} \rangle = \frac{1}{2\mu_0} (\bar{E} \times \bar{B}^*)$$

This removes the minus sign from
the expression for Y , thus ($\bar{B}^* = -\bar{B}$)

$$Y = \sqrt{\frac{\epsilon_0}{\mu_0}} \omega^2$$

d) Let us first calculate the integral U

$$U = \int I_0 \frac{e^{ikz} + e^{-ikz}}{2} e^{-ikz \cos \theta} dz$$

where we have used that $\vec{r} \cdot \vec{r}' = z \cos \theta$
for points (source points) along the z -axis.

$$\begin{aligned} U &= \frac{I_0}{2} \left[\int_{-\lambda/4}^{\lambda/4} e^{ikz(1-\cos \theta)} dz + \int_{-\lambda/4}^{\lambda/4} e^{-ikz(1+\cos \theta)} dz \right] \\ &= \frac{I_0}{2} \left[\frac{e^{i\frac{\pi}{2}(1-\cos \theta)} - e^{-i\frac{\pi}{2}(1-\cos \theta)}}{ik(1-\cos \theta)} + \frac{e^{-i\frac{\pi}{2}(1+\cos \theta)} - e^{i\frac{\pi}{2}(1+\cos \theta)}}{-ik(1+\cos \theta)} \right] \\ &= I_0 \left[\frac{\sin \frac{\pi}{2}(1-\cos \theta)}{k(1-\cos \theta)} + \frac{\sin \frac{\pi}{2}(1+\cos \theta)}{k(1+\cos \theta)} \right] \\ &= I_0 \left[\frac{\cos(\frac{\pi}{2} \cos \theta)}{k(1-\cos \theta)} + \frac{\cos(\frac{\pi}{2} \cos \theta)}{k(1+\cos \theta)} \right] \\ &= \frac{I_0}{k} \cos\left(\frac{\pi}{2} \cos \theta\right) \frac{2}{1-\cos^2 \theta} \\ &= \frac{I_0}{k} \cos\left(\frac{\pi}{2} \cos \theta\right) \frac{2}{\sin^2 \theta} \end{aligned}$$

Thus $\vec{A} = \frac{\mu_0}{2\pi k} I_0 \frac{\cos(\frac{\pi}{2} \cos \theta)}{\sin^2 \theta} \cdot \frac{e^{i(\omega t - kr)}}{r}$

This leads to a time averaged Poynting vector $(\vec{r} \times \vec{A})^2 = \sin^2 \theta \cdot A^2$

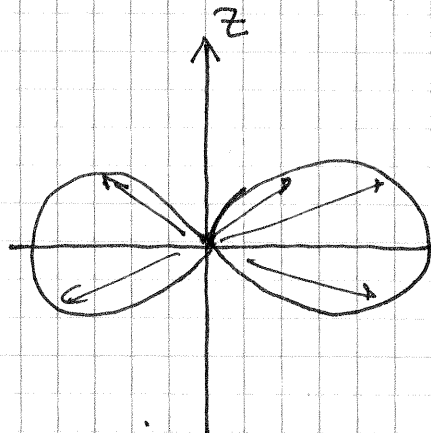
$$\langle \vec{S} \rangle = \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \omega^2 \left(\frac{\mu_0 c}{2\pi \omega} \right)^2 I_0^2 \left(\frac{\cos(\frac{\pi}{2} \cos \theta)}{\sin^2 \theta} \cdot \sin \theta \right)^2 \frac{1}{r^2}$$

$$= \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{I_0^2}{4\pi^2 r^2} \left(\frac{\cos(\frac{\pi}{2} \cos \theta)}{\sin \theta} \right)^2$$

and $\frac{dP}{d\Omega}$

$$\frac{dP}{d\Omega} = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{I_0^2}{8\pi^2} \left(\frac{\cos(\frac{\pi}{2} \cos \theta)}{\sin \theta} \right)^2$$

The radiation pattern.



This pattern is very similar to the short dipole pattern which goes as $\sin^2 \theta$