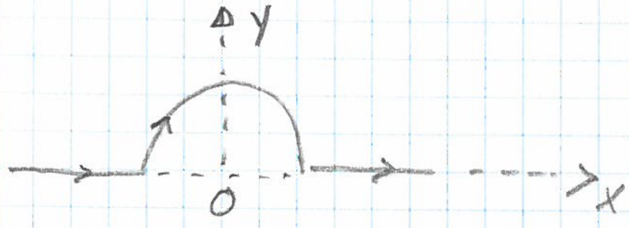
SOLUTION EXAM TFY 4240 Dec 2008

Problem 1



a) The field from a current element $I d\vec{l}'$ is given by Biot-Savart's law

$$d\vec{H} = \frac{1}{4\pi} \frac{I d\vec{l}' \times \hat{r}}{r^2}$$

At position 0, there is no contribution to the magnetic field from the straith parts of the wire since $d\vec{l}' \times \hat{r} = 0$.

Hence the total magnetic field at 0 becomes (where the integration is over the semi-circle)

$$\vec{H} = \int d\vec{H}' = \hat{z} \frac{I}{4\pi R} \int_0^{\pi} d\theta = \underline{\underline{\frac{I}{4R} \hat{z}}}$$

Have here used $dl = R d\theta$.

b) $$|\vec{H}| = \frac{I}{4R} = 0.25 \; 10^2 \; A/m = \underline{\underline{25 \; A/m}}$$

The direction of the field is <u>out</u> of the paper-plane.

Problem 2

a) Ohm's law reads : $\vec{J} = \sigma \vec{E}$

Taking the curl of Faraday's law and combinding it with Ampere's law give:

$$\nabla \times (\nabla \times \vec{E}) = -\partial_t (\nabla \times \vec{B})$$

$$\nabla \underbrace{(\nabla \cdot \vec{E})}_{0} - \nabla^2 E = -\mu \, \partial_t \underbrace{(\vec{J} + \partial_t \vec{D})}_{\sigma \vec{E} + \varepsilon \partial_t \vec{E}}$$

Hence, it follows:

$$\nabla^2 \vec{E} - \mu\varepsilon \partial_t^2 \vec{E} - \mu\sigma \vec{E} = 0 \qquad (2.1)$$

b) With $\vec{E}(\vec{r},t) = \vec{E}_0(\vec{r}) e^{-i\omega t}$ it follows that

$$\partial_t \vec{E}(\vec{r},t) = -i\omega \vec{E}(\vec{r},t)$$

$$\partial_t^2 \vec{E}(\vec{r},t) = -\omega^2 \vec{E}(\vec{r},t)$$

Subsituting these results into Eq. (2.1) gives:

$$[\nabla^2 \vec{E}_0 + \mu\varepsilon\omega^2 \vec{E}_0 + i\sigma\mu\omega \vec{E}_0] e^{-i\omega t} = 0$$

or

$$\nabla^2 \vec{E}_0 + \mu \underbrace{\left(\varepsilon + \frac{i\sigma}{\omega}\right)}_{\epsilon(\omega)} \omega^2 \vec{E}_0 = 0 \qquad (2.2)$$

Hence

$$\epsilon(\omega) = \varepsilon + \frac{i\sigma}{\omega} = \underline{\underline{\varepsilon\left(1 + \frac{i\sigma}{\varepsilon\omega}\right)}}$$

c) Now when $\vec{E}(\vec{r},t) = \vec{E}_0 e^{ikz - i\omega t}$

$$\nabla^2 \vec{E}(\vec{r},t) = -k^2 \vec{E}(\vec{r},t)$$

Hence it follows from Eq. (2.2) that the disp. relation is

$$-k^2 + \mu(\varepsilon + \frac{i\sigma}{\omega})\omega^2 = 0$$

or

$$k = \sqrt{\mu\,\epsilon(\omega)}\,\omega = \underline{\underline{\omega\sqrt{\varepsilon\mu}\sqrt{1 + \frac{i\sigma}{\varepsilon\omega}}}}$$

d) For a good conductor $\sigma/\varepsilon\omega \gg 1$, so that

$$\begin{aligned} k &\simeq \omega\sqrt{\varepsilon\mu}\sqrt{\frac{i\sigma}{\varepsilon\omega}} \\ &= \sqrt{i\,\omega\mu\sigma} \\ &= \frac{1+i}{\sqrt{2}}\sqrt{\omega\mu\sigma} \quad ; \quad \sqrt{i} = \frac{1+i}{\sqrt{2}} \\ &= (1+i)\sqrt{\frac{\omega\mu\sigma}{2}} \end{aligned}$$

With $k = k_1 + ik_2$ it follows that:

$$\underline{\underline{k_1 = k_2 = \sqrt{\frac{\omega\mu\sigma}{2}}}}$$

e) With a plane wave we have for complex k

$$e^{ikz} = e^{+ik_1 z}\, e^{-k_2 z}$$

which is exp. decaying.

The decay constant is $\delta = 1/k_2 = \sqrt{\frac{2}{\omega\mu\sigma}}$

Problem 3

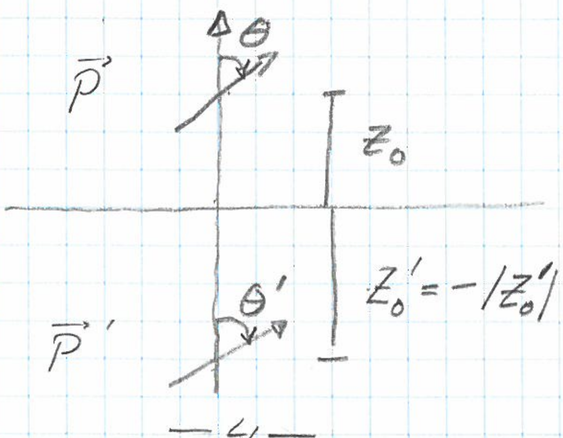
a) For derivation see the book or lecture notes. The meaning of $\vec{R}$ is the distance from the center of the dipole to the observation point.

b) The boundary conditions for V follow from the BC for the fields $\vec{E}$ and $\vec{D}$. They are:

1) $V_1 = V_2$ (from $\vec{E}$)
2) $\varepsilon_1 \partial_n V_1 = \varepsilon_2 \partial_n V_2$ (from $\vec{D}$)

The method of images consists of placing image charges <u>outside</u> the region of interest so that the appropriate BC are satisfied.

c) Since the metal is grounded $V|_{z=0} = 0$. For symmetry reasons, the image dipole should be located on the z-axis, and its orrientation should be so that $V|_{z=0} = 0$ for the sum of the potentials.
Hence the configuration that we consider is as follows, where the angle θ' and pos. z_0' need to be determined.



$\vec{p} = p_0(\sin\theta, 0, \cos\theta)$

$\vec{p}\,' = p_0(\sin\theta', 0, \cos\theta')$

Note: For a dipole located at $\vec{r}_0 = (0,0,z_0)$ one has

$$R = \left[x^2 + y^2 + (z-z_0)^2\right]^{1/2}$$

Hence the total potential at a point (x,y,z) from the two dipoles become:

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0} \frac{p\left[x\sin\theta + (z-z_0)\cos\theta\right]}{\left[x^2+y^2+(z-z_0)^2\right]^{3/2}} + \frac{1}{4\pi\varepsilon_0} \frac{p'\left[x\sin\theta' + (z-z_0')\cos\theta'\right]}{\left[x^2+y^2+(z-z_0')^2\right]^{3/2}} \quad (3.1)$$

Now ~~from~~ the boundary condition says that at $z=0$ (and for all x and y) the potential is zero.

Therefore the spatial decay must be the same i.e. $|z_0| = |z_0'|$ or

$$z_0' = -z_0 \quad (3.2)$$

Moreover, since x and y are independent we must have:

$$\left.\begin{aligned} p\sin\theta &= p'\sin\theta' \\ p\cos\theta &= -p'\cos\theta' \end{aligned}\right\} \Rightarrow \tan\theta = -\tan\theta'$$

Hence:

$$\begin{aligned} p' &= p \\ \theta' &= -\theta \end{aligned} \quad (3.3)$$

and the final expression for V follows from (3.1).

d) The induced surface charge follows from the continuity of the normal comp. of $\vec{D}$.

$$\sigma = D_n\big|_{z=0} = -\varepsilon_0 \partial_z V\big|_{z=0}$$

$$= -\frac{p\cos\theta}{2\pi\left[x^2+y^2+z_0^2\right]^{3/2}}$$

$$+ \frac{3pz_0(-x\sin\theta + z_0\cos\theta)}{2\pi\left[x^2+y^2+z_0^2\right]^{5/2}}$$

Problem 4

a) Since it is a circular path it follows

$$\phi(t) = \omega t \qquad \text{(assuming } \phi(0)=0\text{)}$$

and

$$\vec{r}_p(t) = R(\cos\omega t, \sin\omega t, 0)$$

$$\vec{v}_p(t) = \frac{d}{dt}\vec{r}_p(t) = r_0\omega(-\sin\omega t, \cos\omega t, 0$$

$$\vec{a}_p(t) = \frac{d}{dt}\vec{v}_p(t) = r_0\omega^2(-\cos\omega t, -\sin\omega t, 0)$$

$$= -\omega^2\vec{r}_p(t) \qquad (4.1)$$

According to the Lorentz force the magnetic field is pointing downwards in order for the force to point inwards.
It is thus perpendicular to $\vec{v}_p(t)$.

b) $\hat{R}(t)$ is the distance between the particle and the observer, i.e. $\hat{R}(t) \propto \vec{r} - \vec{r}_p(t)$

It is the retarded time that should be used.

c) We want to calculate

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{4\pi c^3} \left\langle (\hat{R} \times \vec{a}_p)^2 \right\rangle \tag{4.2}$$

Since $r \ll r_0$ one has

$$\hat{R}(t) = \frac{\vec{r} - \vec{r}_p(t)}{|\vec{r} - \vec{r}_p(t)|} \simeq \hat{r}$$

which is time-independent. Therefore, with Eq. (4.1):

$$\begin{aligned} \hat{R} \times \vec{a}_p &= -\omega^2 \hat{R} \times \vec{r}_p \\ &= -\omega^2 r_0 \cos(\omega t)\, \hat{R} \times \hat{e}_x - \omega^2 r_0 \sin(\omega t)\, \hat{R} \times \hat{e}_y \end{aligned}$$

Taking the square of this expression:

$$\begin{aligned} (\hat{R} \times \vec{a}_p)^2 = \omega^4 r_0^2 \Big[& \cos^2(\omega t) \; (\hat{R} \times \hat{e}_x)^2 \\ & + \sin^2(\omega t) \; (\hat{R} \times \hat{e}_y)^2 \\ & + \underbrace{2\cos(\omega t)\sin(\omega t)}_{\sin(2\omega t)} \; (\hat{R} \times \hat{e}_x) \cdot (\hat{R} \times \hat{e}_y) \Big] \end{aligned}$$

Hence the time-average becomes:

$$\left\langle (\hat{R} \times \vec{a}_p)^2 \right\rangle = \frac{\omega^4 r_0^2}{2} \left[(\hat{R} \times \hat{e}_x)^2 + (\hat{R} \times \hat{e}_y)^2 \right] \tag{4.3}$$

where we used that.

$$\langle \cos^2 \omega t \rangle = \langle \sin^2 \omega t \rangle = 1/2$$

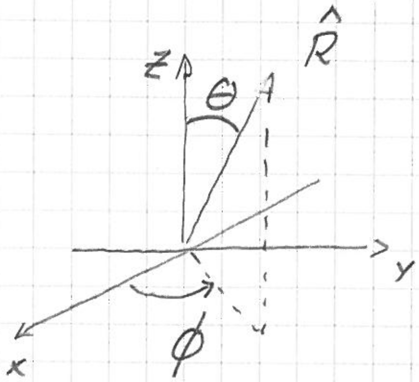
$$\langle \sin \omega t \rangle = 0.$$

One now needs to calculate $(\hat{R}\times\hat{e}_x)^2$ and $(\hat{R}\times\hat{e}_y)^2$.
We start by calculating:

$$
\begin{aligned}
(\hat{a}\times\hat{b})^2 &= (\hat{a}\times\hat{b})_i(\hat{a}\times\hat{b})_i \\
&= \varepsilon_{ijk}\, a_j b_k \;\varepsilon_{ilm}\, a_l b_m \\
&= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\; a_j a_l \; b_k b_m \\
&= \underbrace{a_j a_j}_{1}\;\underbrace{b_k b_k}_{1} - a_j b_j\; a_k b_k \\
&= 1 - (\hat{a}\cdot\hat{b})^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
\langle(\hat{R}\times\vec{a}_p)^2\rangle &= \frac{\omega^4 r_0^2}{2}\left[1-(\hat{R}\cdot\hat{e}_x)^2 + 1 - (\hat{R}\cdot\hat{e}_y)^2\right] \\
&= \frac{\omega^4 r_0^2}{2}\left[2 - \sin^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi\right] \\
&= \frac{\omega^4 r_0^2}{2}\left[2 - \sin^2\theta\right] \\
&= \frac{\omega^4 r_0^2}{2}\left[1 + \cos^2\theta\right]
\end{aligned}
$$



and

$$
\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{8\pi c^3}\,\omega^4 r_0^2\,(1+\cos^2\theta)
$$

The expression is independent of ϕ since on average the apparent particle movement is the same for all observation point $O = (r, \theta, \phi)$.

The assumption $r \ll r_0$ is a significant simpl. since then $\hat{R}$ is time-independent.

d) The total radiated power:

$$\int d\Omega \, (1+\cos^2\theta) = \int_0^{2\pi} d\phi \int_{-1}^{1} dx \, (1+x^2)$$

$$= 2\pi \left[x + \frac{x^3}{3} \right]_{-1}^{1} = \frac{16\pi}{3}$$

$$P = \int d\Omega \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{4\pi\varepsilon_0} \frac{2q^2}{3c^3} \omega^4 r_0^2 \qquad (4.4)$$

Eq (4.4) is Larmor's formula with $a = \omega^2 r_0$ (as expected).

e) Let us assume that $v_p \ll c$ so that its kinetic energy is:

$$K = \frac{1}{2} m v_p^2 = \frac{1}{2} m r_0^2 \omega^2$$

Hence the ratio between the radiated power and the initial kinetic energy is

$$\frac{P}{K} = \frac{1}{4\pi\varepsilon_0} \frac{4q^2}{3mc^3} \omega^2$$

For instance for an electron, with $\omega \sim 1$, this ratio is essentially zero

$$\frac{P}{K} \sim 10^{-23}.$$

So even if the particle is radiating, its energy (taken from its kinetic energy) is practically constant.

After about 10^6 years ($= 10^{13}$ s) its radiated energy is still 10^{-10} of its initial kinetic energy.

In addition to this, there are energy corrections that we have neglected here. coming from the radiative reaction.

NOTE: H is <u>not</u> doing work on the particle