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Solution to the exam in TFY4240 Electromagnetic Theory

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This solution consists of 6 pages.

Problem 1. Potential theory

a) The formulas are given in the appendix:

$$\mathbf{E} = -\nabla V - \partial_t \mathbf{A} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

We immediately observe that adding a constant to either potential will not change the fields, as they depend on the potentials through derivatives. More generally, one may perform a *gauge transformation* on the potentials, that is, one can change the potentials as one wishes, as long as the *fields remain unchanged*. Consult Griffiths, Chapter 10.1.3, for details.

b) Gauss' law reads $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$. Inserting the expression for $\mathbf{E}$ and taking the gradient of this equation yields

$$\nabla \cdot (-\nabla V - \partial_t \mathbf{A}) = \frac{\rho}{\varepsilon_0}, \text{ or}$$
$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} - \partial_t \nabla \cdot \mathbf{A}$$

Using the hint, adding a term on each side of the equation, we get

$$\nabla^2 V - \frac{1}{c^2}\partial_t^2 V = -\frac{\rho}{\varepsilon_0} - \partial_t \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\partial_t V \right),$$

which is exactly what we were asked to show.

c) Following the hint, we first derive the wave equation for $\mathbf{A}$. Inserting the expressions for $\mathbf{E}$ and $\mathbf{B}$ in terms of potentials into Ampère's law, we get

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \partial_t (-\nabla V - \partial \mathbf{A}).$$

Using the formula $\nabla \times (\nabla \times \mathbf{A}) \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, we tidy up and get

$$\nabla^2 \mathbf{A} - \varepsilon_0 \mu_0 \partial_t^2 \mathbf{A} = -\mu_0 \mathbf{J} + \nabla \left(\varepsilon_0 \mu_0 \partial_t V + \nabla \cdot \mathbf{A} \right).$$

Finally, we notice that the expression in the parenthesis are identical to 0 in the Lorentz gauge, yielding the two following symmetrical wave equations for V and $\mathbf{A}$:

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\partial_t^2\mathbf{A} = -\mu_0\mathbf{J} \quad \text{and}$$
$$\nabla^2 V - \frac{1}{c^2}\partial_t^2 V = -\frac{\rho}{\varepsilon_0}.$$

Problem 2. The image method

a) The method of images consists of replacing the electrostatic problem we are trying to solve with another so-called equivalent problem. The equivalent problem's solution should satisfy the same boundary conditions as the original electrostatic problem. Due to the theorem of uniqueness and its corollary, this solution is thus the same solution as the solution to our original problem.

As the metal plane is grounded, the potential has to be zero everywhere on the surface of the metal plane:

$$V|_{z=-a} = 0.$$

In addition, the normal derivative of the potential is discontinuous at the interface, due to the discontinuity of $\mathbf{D}$:

$$\mathbf{D}^{\perp}|_{z=-a} = \sigma_f, \text{ or}$$
$$\varepsilon_0\partial_z V|_{z=-a} = \sigma_f.$$

Here, σ_f is the free surface charge density.

b) This problem is solved in Griffiths, Chapter 3.2.1. After translating the coordinate system, the solution reads:

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\left\{\left[x^2+y^2+z^2\right]^{-1/2} - \left[x^2+y^2+(z+2a)^2\right]^{-1/2}\right\}.$$

Inserting $z=-a$, we easily see that the two terms cancel, and that $V(z=-a)=0$.

c) This problem can seem tricky, but it is not so hard if you are systematic. We simply need to add up the contribution from an infinite number of point charges. The "zeroth" term stems from the original point charge at the origin. Then, the next term stems from two "mirror charges;" one for each plane. But, each of these mirror charges need a mirror charge for the opposite plane... and so on. Continuing this summation gives us positive and negative charges at:

$$q \text{ at } z = 4na, \qquad n = 0, \pm1, \pm2, \ldots$$
$$-q \text{ at } z = (4n+2)a, \qquad n = 0, \pm1, \pm2, \ldots$$

The analytical expression for V is somewhat involved and it was not a requirement to state this on the exam. It is simply found by summing up the contributions from all the point charges.

d) This is done by taking the normal derivative of the potential at the interface and using the discontinuity of the normal derivative of the field, $\partial_z V|_{z=-a}$. Griffiths does this for the "simple" image problem in Chapter 3.2.1.

Problem 3. TEM modes in coaxial waveguides

a) If the electric field near a perfect conductor has a component parallel to the surface, it would immediately create an electric current in the same direction as the parallel component of the field. This current would run until the field has been cancelled, i.e. charges will rearrange in such a way that there are no parallel components on the electric field. Hence, the paralallel component must be 0. The same argument goes for the inside of the conductor.

b) Inside the conductor, there is no electric field, according to the previous problem. Inserting $\mathbf{E} = 0$ into Faraday's law, we see that the time derivative of the magnetic field must be 0 inside the conductor. If the magnetic field was 0 at $t = -\infty$, it must be 0 now as well. Hence, it is reasonably to assume that the magnetic field inside a perfect conductor is 0. (This can in fact be proven more rigorously.) As $B^{\perp}$ is continuous across an interface, and $\mathbf{B} = 0$ inside the conductor, we must have that $B^{\perp} = 0$ at the interface.

c)

d) These two problems are inexorably intertwined; hence, they will be solved together.

First, we notice that $\mathbf{E}$ depends on z and t only through an exponential function, and that it does not depend on ϕ at all. This means that we can write $\partial_z \to \mathrm{i}\beta$, $\partial_t \to -\mathrm{i}\omega$ and $\partial_\phi = 0$.

$\mathbf{E}$ only has a radial component. From the formula for the divergence in cylindrical coordinates, we easily see that the divergence of $\mathbf{E}$ vanishes:

$$\nabla \cdot \mathbf{E} \propto \partial_r \left(r E_r(r) \right) = 0.$$

Hence, the two Gauss' laws are fulfilled.

Faraday's law is a bit more involved, but easily handled using our results for the partial derivatives. First of all, we notice that the right hand side of Faraday's law is $-\partial_t \mathbf{B} = \mathrm{i}\omega\mathbf{B}$. Second, we notice that only the $\hat{\boldsymbol{\phi}}$ component of $\nabla \times \mathbf{E}$ survives. This turns out to be

$$\nabla \times \mathbf{E} = \partial_z (\mathbf{E} \cdot \hat{\mathbf{r}}) \hat{\boldsymbol{\phi}} = \mathrm{i}\beta E_r(r) \mathrm{e}^{\mathrm{i}(\beta z - \omega t)}.$$

Combining the left and the right hand side, we obtain

$$\mathrm{i}\beta E_r(r) \mathrm{e}^{\mathrm{i}(\beta z - \omega t)} \hat{\boldsymbol{\phi}} = \mathrm{i}\omega \mathbf{B}, \text{ or}$$

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\beta}{\omega} E_r(r) \mathrm{e}^{\mathrm{i}(\beta z - \omega t)}.$$

Ampère's law gives essentially the same result, except now, some additional constants appear on the scene. Assuming that both Faraday's and Ampères laws hold, the constants must be equal, or more precisely,

$$\frac{\beta^2}{\omega^2} = \varepsilon_r \varepsilon_0 \mu_0.$$

This is the dispersion relation. The phase and group velocities are equal, as the wave number is simply a constant times the frequency:

$$\beta = \sqrt{\varepsilon_r \varepsilon_0 \mu_0}\,\omega$$

The phase and group velocities are proportional to $\varepsilon_r^{-1/2}$. This is a very simple dispersion relation, very advantageous for signal propagation, and the group velocity is as high as we could reasonably expect (same as a plane wave in the dielectric in question).

e) We solve this problem by first determining the time-averaged Poynting's vector, and then integrating it over the cross-section of the cable where we have fields present. Recalling the complex-notation formula for the time-averaged Poynting's vector, we find that:

$$\langle \mathbf{S} \rangle = \frac{1}{2\mu_0} \mathbf{E} \times \mathbf{B}^* = \hat{\boldsymbol{z}} \sqrt{\frac{\varepsilon_r \varepsilon_0}{4\mu_0}} \left(\frac{aE_0}{r} \right)^2 .$$

Now, we integrate the Poynting's vector over the cable cross-section to obtain the total transmitted power:

$$\int_{r=a}^{b} \langle \mathbf{S} \rangle \cdot \mathrm{d}\mathbf{A} = \int_{r=a}^{b} 2\pi r \langle \mathbf{S} \rangle \cdot \hat{\boldsymbol{z}}\, \mathrm{d}r = \pi \frac{\varepsilon_r \varepsilon_0}{\mu_0} a^2 E_0^2 \int_{r=a}^{b} r^{-1}\, \mathrm{d}r.$$

The integral can now easily be found:

$$\int_{r=a}^{b} r^{-1}\, \mathrm{d}r = \ln r|_{r=a}^{b} = \ln \frac{b}{a}.$$

Finally, the total transmitted power through a cable cross-section is equal to

$$P = \pi \frac{\varepsilon_r \varepsilon_0}{\mu_0} a^2 E_0^2 \ln \frac{b}{a}.$$

The units can be checked using the information found in the formula appendix, where the units of various constants are given.

$$\left[\sqrt{\varepsilon_0/\mu_0}\, a^2 E_0^2 \right] = \sqrt{\frac{\mathrm{C}^2/\mathrm{Nm}^2}{\mathrm{N}/\mathrm{A}^2}}\, \mathrm{m}^2 \left(\frac{\mathrm{J}}{\mathrm{Cm}} \right)^2 = \mathrm{J/s} = \mathrm{W}.$$

(Recall that A = C/s and J = Nm in the above calculation.) This is the unit of power, which is correct.

Problem 4. EM waves in a plasma

a) Working in Fourier space (think of a plane wave $\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)}$), we know that $\nabla^2 = -k^2$ and $\partial_t^2 = -\omega^2$. Inserting this and the expression for ε_r into the wave equation, we get

$$-k^2 + \frac{\omega^2 - \omega_p^2}{c^2} = 0$$

which can easily be manipulated to be on the same form as the expression given in the exam problem.

b) We find k using the dispersion relation found above:

$$k^2 = \frac{\omega^2 - \omega_p^2}{c^2}.$$

We see that $\omega < \omega_p \Rightarrow k^2 < 0$. A complex wave number gives an exponentially decaying wave, that is, the wave does not propagate if $\omega < \omega_p \Rightarrow k^2 < 0$.

c) Using implicit derivation with respect to k, we see that

$$2\omega \frac{\mathrm{d}\omega}{\mathrm{d}k} = 2c^2 k.$$

Using the fact that $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}$, we find that

$$v_g(k) = \frac{c^2 k}{\omega} = \frac{c^2 k}{\sqrt{\omega_p^2 + c^2 k^2}}.$$

Note that the group velocity must be 0 if $\omega < \omega_p$, as waves no longer propagate in this case. As $k = 0$ when $\omega = \omega_p$, we see that the expression for $v_g(k)$ is consistent with the dispersion relation.

d) This problem can be solved by matching boundary conditions at the interface between the vacuum and the plasma. We consider a plane wave traveling along the x axis, normally incident on a plasma filling the half-space $x > 0$. Because we have normal incidence, $\mathbf{E}$ and $\mathbf{B}$ are both parallel to the interface, that is, they both lie in the yz plane. Relevant boundary conditions (which should be derived if there is any doubt!) are that $\mathbf{E}^{\parallel}$ and $\mathbf{B}^{\parallel}$ are both continuous, because we have no surface currents in the plasma (this can be seen from the wave equation, which contains no $\mathbf{J}$ term).

First, we find the wave number in the plasma. This is given from the result in the first part of the problem:

$$k = \frac{\sqrt{\omega^2 - \omega_p^2}}{c} = \omega \frac{\sqrt{1 - \omega_p^2/\omega^2}}{c} = \sqrt{\varepsilon_r}\frac{\omega}{c}.$$

Assume that we have an incident and a reflected plane wave in the half-space $x < 0$. The field there is thus

$$\mathbf{E} = \hat{\mathbf{x}} E_I \mathrm{e}^{\mathrm{i}(kx - \omega t)} + \hat{\mathbf{x}} E_R \mathrm{e}^{\mathrm{i}(-kx - \omega t)}.$$

In the plasma, the field consists only of the transmitted wave:

$$\mathbf{E} = \hat{\mathbf{x}} E_T \mathrm{e}^{\mathrm{i}(k_p x - \omega t)}.$$

In these expressions, E_I is the amplitude of the incident field at $x = 0$, and E_R and E_T are the amplitudes of the reflected and transmitted fields at $x = 0$, respectively.

With B_I, B_R, and B_T being the amplitudes of incident, reflected and transmitted magnetic fields at $x = 0$, the boundary conditions read

$$\begin{aligned} E_I + E_R &= E_T \quad \text{and} \\ B_I - B_R &= B_T. \end{aligned}$$

The sign "flip" in the second boundary condition comes from the reflection of the electromagnetic wave, which reflects the right hand system formed by $\mathbf{k}, \mathbf{E},$ and $\mathbf{B}$.

By virtue of Faraday's law, the electric and magnetic field components are coupled:

$$\partial_x E_y = -\partial_t B_z \Rightarrow \mathrm{i}kE_y = \mathrm{i}\omega B_z \Rightarrow E_y = \frac{\omega}{k} B_z.$$

(E_y and B_z are the only components of the electric and magnetic fields in our chosen coordinate system.) This and the dispersion relation combines into

$$E_I - E_R = \sqrt{\varepsilon_r} E_T,$$

where we recall that $\varepsilon_r = 1$ in vacuum.

Dividing both of the boundary condition equations by E_I and recalling that $r = E_R/E_I$ and $t = E_T/E_I$, we obtain two equations for the reflection and transmission amplitudes:

$$1 + r = t,$$
$$1 - r = \sqrt{\varepsilon_r} t.$$

A linear system with 2 unknowns can be solved by hand. In this case, we get the solution:

$$r = \frac{1 - \sqrt{\varepsilon_r}}{1 + \sqrt{\varepsilon_r}},$$
$$t = \frac{2}{1 + \sqrt{\varepsilon_r}}.$$

As $\sqrt{\varepsilon_r} = \sqrt{1 - \omega_p^2/\omega^2}$, we see that $\sqrt{\varepsilon_r}$ is purely imaginary when $\omega < \omega_p$. Because the reflection coefficient is $R = |r|^2 = rr^*$, we see that when this is the case,

$$R = rr^* = \frac{1 - \sqrt{\varepsilon_r}}{1 + \sqrt{\varepsilon_r}} \frac{1 + \sqrt{\varepsilon_r}}{1 - \sqrt{\varepsilon_r}} = 1.$$

This means that when the waves do not propagate inside the plasma, the reflection coefficient is 1, i.e. all the energy is reflected at the plasma interface. We have a consistent result!