

Problem 1

a) The current density is given by

$$\vec{J}(\vec{r}) = \frac{\hat{z} I}{\pi(b^2 - a^2)} \theta(r_{||} - a) \theta(b - r_{||}) \quad (1-1)$$

b) The problem is a magnetostatic prob. By symmetry it follows that

$$\vec{H} \propto \hat{\phi}$$

and the magnitude of the H-field follows from Amperes law by applying it to a circular Amperian loop of radius $R_{||}$ and placed in the xy-plane

$$\nabla \times \vec{H} = \vec{J}$$

$$\int d\vec{A} \cdot \nabla \times \vec{H} = \int d\vec{A} \cdot \vec{J}$$

$$\oint d\vec{e} \cdot \vec{H} = 2\pi R_{||} H = \int d\vec{A} \cdot \vec{J}$$

$$\Rightarrow H(R_{II}) = \frac{1}{2\pi R_{II}} \int_0^{R_{II}} dr_{II} r_{II} \int_0^{2\pi} d\phi J(r_{II})$$

$$= \frac{1}{R_{II}} \frac{I}{\pi(b^2 - a^2)} \int_a^{R_{II}} dr_{II} r_{II} \Theta(b - r_{II})$$

i) $R_{II} < a$: $H(R_{II}) = 0$ (since $\vec{J} = 0$)

ii) $a < R_{II} < b$

$$H(R_{II}) = \frac{1}{R_{II}} \frac{I}{\pi(b^2 - a^2)} \frac{1}{2} (R_{II}^2 - a^2)$$

$$= \frac{I}{2\pi R_{II}} \frac{R_{II}^2 - a^2}{b^2 - a^2}$$

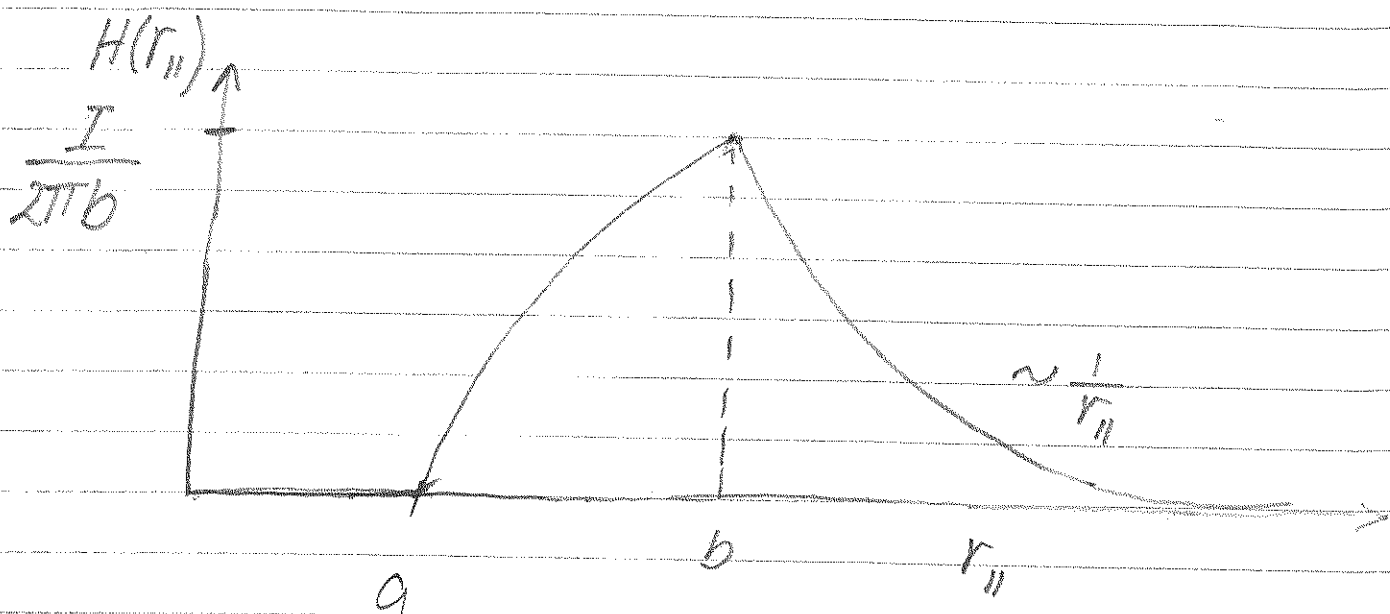
iii) $R_{II} > b$:

$$H(R_{II}) = \frac{I}{2\pi R_{II}}$$

Renaming $R_{II} \rightarrow r_{II}$ it follows

$$H(\vec{r}_{II}) = \begin{cases} 0 & r_{II} < a \\ \frac{I}{2\pi r_{II}} \frac{r_{II}^2 - a^2}{b^2 - a^2} & a < r_{II} < b \\ 1 & r_{II} > b \end{cases}$$

Sketch



c) Since this is a magnetostatic problem $\vec{E}(\vec{r}) = 0$ everywhere. This follows since the source of the electric field, $\rho(\vec{r})$, is zero everywhere.

Problem 2

a) We need to solve Laplace eq. which is a separable PDE. Since the problem has azimuthal symmetry the general solution of LE is given by Eq. (1).

Terms of the form r^l and $1/r^{l+1}$ appear in this eq. since they are the independent solutions of the radial eq. $P_l(\cos\theta)$ denotes the Legendre polynomials.

b) The boundary conditions on $V(\vec{r})$ are

(i) $r=0$: $V(\vec{r})$ is finite (2-1)
(ii) $r \rightarrow \infty$: $V(\vec{r})$ corresponds to $\vec{E}_0 = -\nabla V$

$$\begin{aligned} \Rightarrow V(\vec{r}) &= -E_0 z \\ &= -E_0 r \cos\theta \\ &= -E_0 r P_1(\cos\theta) \end{aligned} \quad (2-2)$$

(iii) $r=a$: $V^{(1)}(\vec{r})|_{r=a} = V^{(2)}(\vec{r})|_{r=a} \quad (2-3)$

$$\epsilon_1 \left[\frac{\partial}{\partial r} V^{(1)}(\vec{r}) \right] \Big|_{r=a} = \epsilon_2 \left[\frac{\partial}{\partial r} V^{(2)}(\vec{r}) \right] \Big|_{r=a} \quad (2-4)$$

This means that $V(\vec{r})$ and $\epsilon \frac{\partial}{\partial r} V(\vec{r})$ are continuous over the surface.

Here $V^{(n)}(\vec{r})$ denotes the potential in region n .

c) We start by writing the potential in the two regions in the general form

$$V^{(1)}(\vec{r}) = \sum_{l=0}^{\infty} \left[A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\theta) \quad (2-5)$$

and

$$V^{(2)}(\vec{r}) = \sum_{l=0}^{\infty} \left[C_l r^l + \frac{D_l}{r^{l+1}} \right] P_l(\cos\theta) \quad (2-6)$$

Now from BC (2-1) it follows $B_l = 0$ for all l 's. Moreover, from (2-2) we find

$$C_l = \begin{cases} -E_0 & l=1 \\ 0 & l>1 \end{cases} \quad (2-7)$$

From the BC on the surface of the sphere one gets:

$$\begin{aligned} \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) \\ = -E_0 a P_1(\cos\theta) + \sum_{l=0}^{\infty} \frac{D_l}{a^{l+1}} P_l(\cos\theta) \end{aligned} \quad (2-8)$$

and

$$\epsilon_1 \sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos\theta) \quad (2-9)$$

$$= -\epsilon_2 E_0 P_1(\cos\theta) - \epsilon_2 \sum_{l=0}^{\infty} (l+1) \frac{D_l}{a^{l+2}} P_l(\cos\theta)$$

which should be satisfied for any value of θ .

Now introducing $x = \cos\theta$ and operating from the left with the operator

$$\int_{-1}^1 dx P_m(x)$$

on these relations gives after using the orthogonality condition

$$\int_{-1}^1 dx P_m(x) P_n(x) = \frac{2}{2m+1} \delta_{mn}$$

$$A_m a^m = -E_0 a \delta_{m1} + D_m a^{-(m+1)}$$

$$E_1 m A_m a^{m-1} = -E_2 E_0 \delta_{m1} - E_2 (m+1) D_m a^{-(m+2)}$$

or

$$\begin{bmatrix} a^m & -a^{-(m+1)} \\ E_1 m a^{m-1} & E_2 (m+1) a^{-(m+2)} \end{bmatrix} \begin{bmatrix} A_m \\ D_m \end{bmatrix}$$

$$= -\delta_{m1} E_0 \begin{bmatrix} a \\ E_2 \end{bmatrix} \quad (2-10)$$

When $m \neq 0$ it follows immediately that

$$A_m = D_m = 0 \quad m \neq 1,$$

However, for $m=1$:

$$\begin{bmatrix} a & -a^{-2} \\ \epsilon_1 & 2\epsilon_2 a^{-3} \end{bmatrix} \begin{bmatrix} A_1 \\ D_1 \end{bmatrix} = -E_0 \begin{bmatrix} a \\ \epsilon_2 \end{bmatrix} \quad (2-11)$$

a system that has the solution

$$\begin{bmatrix} A_1 \\ D_1 \end{bmatrix} = \frac{-a^2 E_0}{\epsilon_1 + 2\epsilon_2} \begin{bmatrix} 2\epsilon_2 a^{-3} & -\epsilon_1 \\ a^{-2} & a \end{bmatrix}^T \begin{bmatrix} a \\ \epsilon_2 \end{bmatrix}$$

$$= \frac{-a^2 E_0}{\epsilon_1 + 2\epsilon_2} \begin{bmatrix} +2\epsilon_2 a^{-2} + \epsilon_2 a^{-2} \\ -\epsilon_1 a + \epsilon_2 a \end{bmatrix}$$

$$= \frac{E_0}{\epsilon_1 + 2\epsilon_2} \begin{bmatrix} -3\epsilon_2 \\ +(\epsilon_1 - \epsilon_2) a^3 \end{bmatrix} \quad (2-12)$$

Therefore, the potential becomes :

$$V(\vec{r}) = \begin{cases} -\frac{3\epsilon_2}{\epsilon_1 + 2\epsilon_2} E_0 r \cos\theta, & r < a \\ -\left[1 - \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} \left(\frac{a}{r}\right)^3\right] E_0 r \cos\theta & r > a \end{cases}$$

$$k_1 = \frac{3\epsilon_2}{\epsilon_1 + 2\epsilon_2}$$

$$k_2 = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2}$$

(2-13)

Alternatively, one could have derived this expression by using that the expressions in front of $P_e(\cos\theta)$ in Eqs. (2-8) and (2-9) have to be the same on both sides of the eqs.

d) The electric field is given by

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r})$$

Inside the sphere one finds the constant field

$$\vec{E}(\vec{r}) = \frac{3\epsilon_2}{\epsilon_1 + 2\epsilon_2} \vec{E}_0, \quad r < a \quad (2-14)$$

To get the field outside the sphere, we start by calculating $\nabla \left(\frac{\vec{E}_0 \cdot \vec{r}}{r^3} \right)$:

$$\partial_i \left(\frac{\vec{E}_0 \cdot \vec{r}}{r^3} \right) = \frac{E_{0,k} (\partial_i r_k) r^3 - \partial_i (r^3) r_k}{r^6}$$

$$= \frac{E_{0,k} \delta_{ik} r^3 - 3r^2 \partial_i (r) r_k}{r^6}$$

$$\partial_i (r) = \partial_i \sqrt{x_m x_m} = \frac{2x_m \delta_{im}}{2r} = \frac{x_i}{r}$$

$$= \frac{E_{0,k} \delta_{ik} r^3 - 3r^2 [\hat{r}]_i r_k}{r^6}$$

$$= \frac{E_{0,i} - 3(\vec{E}_0 \cdot \hat{r}) [\hat{r}]_i}{r^3} \quad (2-15)$$

THUS

$$\nabla \left(\frac{\vec{E}_0 \cdot \vec{r}}{r^3} \right) = \frac{\vec{E}_0 - 3(\vec{E}_0 \cdot \hat{r}) \hat{r}}{r^3}$$

so that the field outside the sphere becomes ($r > a$)

$$\vec{E}(\vec{r}) = \vec{E}_0 + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} a^3 \left[\frac{3(\vec{E}_0 \cdot \hat{r}) \hat{r} - \vec{E}_0}{r^3} \right] \quad (2-16)$$

e) The interpretation of Eq (2-16) is that the first term is the external field while the 2nd term result from the polarization of the sphere that \vec{E}_0 causes inside the sphere. Actually, a dipole moment is induced in the sphere.

The field from an infinitesimal dipole of dipole moment \vec{p} is given by*

$$\frac{1}{4\pi\epsilon_0\epsilon_2} \frac{3(\vec{p} \cdot \hat{r}) \hat{r} - \vec{p}}{r^3}$$

Hence we conclude that the induced dipole moment in the sphere is

$$\vec{p} = 4\pi\epsilon_0 a^3 \epsilon_2 \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} \vec{E}_0 \quad (2-17)$$

We note that $\vec{p} \propto \vec{E}_0$

* Here ϵ_0 is the permittivity of free space

The same conclusion can be reached by noting that the 2nd term of the potential $V(\vec{r})$ for $r > a$ has the form of an electric dipole potential

$$\frac{1}{4\pi\epsilon_0\epsilon_2} \frac{\vec{p} \cdot \hat{r}}{r^2}$$

The polarizability becomes:

$$\alpha = 4\pi\epsilon_0 a^3 \epsilon_2 \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} \quad (2-18)$$

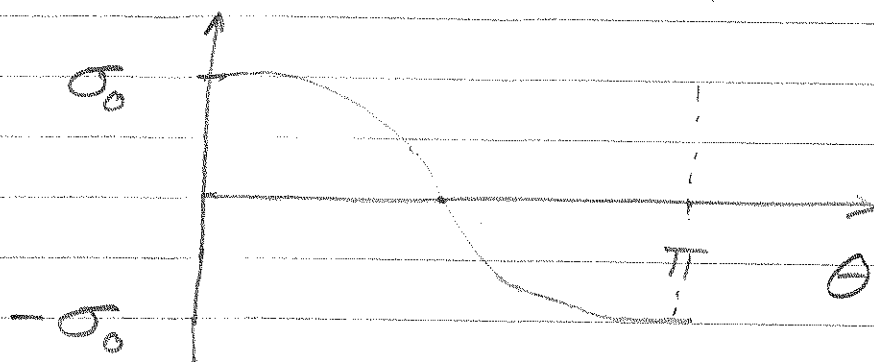
The polarization, \vec{P} , of the sphere is defined as dipole moment per unit volume. Therefore, we have that

$$\vec{P} = \frac{\vec{p}}{\frac{4}{3}\pi a^3} = 3\epsilon_0 \epsilon_2 \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} \vec{E}_0 \quad (2-19)$$

∫] The charge density induced at the surface of the sphere is given by

$$\begin{aligned} \sigma(\theta) &= \hat{r} \cdot \vec{P} \\ &= 3\epsilon_0 \epsilon_2 \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} E_0 \cos\theta \end{aligned}$$

$$\sigma(\theta) = \sigma_0 \cos\theta$$



The positive/negative charge is induced in the same/opposite direction of the field. Maximum/minimum is at $\theta = 0 / \theta = \pi$ and $\sigma(\theta)$ is zero for $\theta = \pi/2$. The solution has axial sym. This behavior is as expected.

9] Since $\sigma(\theta)$ has axial symmetry, i.e., it does not depend on the azimuthal angle ϕ , a rotation of the sphere about \vec{E}_0 will not give rise to a (polarization) current.

Hence, no magnetic field will be produced due to the rotation.

Problem 3

aj In general one has (in the Lorentz gauge)

$$\begin{aligned}
 \vec{A}(\vec{r}, t) &= \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \\
 &= \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{J}(\vec{r}', \omega) e^{-i\omega t_r}}{|\vec{r} - \vec{r}'|} \\
 &= \frac{\mu_0}{4\pi} e^{-i\omega t} \int d^3r' \frac{\vec{J}(\vec{r}', \omega) e^{ik|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} \quad (3-1)
 \end{aligned}$$

In the far field ($kr \gg 1$, $r \gg r'$) we approximate

$$\frac{1}{|\vec{r} - \vec{r}'|} \approx \frac{1}{r} \quad (3-2)$$

$$\begin{aligned}
 k|\vec{r} - \vec{r}'| &= k \sqrt{r^2 + r'^2 - 2\vec{r} \cdot \vec{r}'} \\
 &= kr \sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{\hat{r} \cdot \vec{r}'}{r}} \\
 &\approx kr \left(1 - \frac{\hat{r} \cdot \vec{r}'}{r}\right) \quad (3-3)
 \end{aligned}$$

Eqs (3-2) and (3-3) substituted into (3-1) gives the result that should be demonstrated.

$$\vec{A}(\vec{r}, t) \approx \frac{\mu_0}{4\pi} \frac{e^{i\vec{k}\cdot\vec{r} - i\omega t}}{r} \vec{f}(\theta, \phi)$$

where

$$\vec{f}(\theta, \phi) = \int d^3r' \vec{J}(\vec{r}', \omega) e^{-i\vec{k}\cdot\vec{r}'}$$

b) It follows directly from the geometry that

$$\vec{J}(\vec{r}, t) = J(\vec{r}, t) \hat{\phi}$$

Since the radius of the loop is $r=a$ and it is placed in the xy -plane it follows that in spherical coordinates

$$J(\vec{r}, t) \propto \delta(r-a) \delta(\theta - \pi/2)$$

Moreover, $J(\vec{r}, t)$ must be prop. to $I(t)$ so we write,

$$J(\vec{r}, t) = \alpha I(t) \delta(r-a) \delta(\theta - \pi/2)$$

where α is a constant to be determined. We note that $[\alpha] = m^{-1}$ in order for J to have the correct dimension.

So, how can we determine α ? We know that

$$\int d\vec{A} \cdot \vec{J}(\vec{r}, t) = I(t)$$

By integrating over, for instance, the $x+z$ -plane we get (using polar coordinates)

$$\int_0^{\infty} dr r \int_0^{\pi} d\theta \alpha I(t) \delta(r-a) \delta(\theta-\pi/2) = \alpha a I(t)$$

Since this integral should be equal to $I(t)$ it follows that $\alpha = 1/a$. Hence, we conclude

$$\vec{J}(\vec{r}, t) = \frac{I(t)}{a} \delta(r-a) \delta(\theta-\pi/2) \hat{\phi} \quad (3-4)$$

[The same result can be obtained by e.g. integrating $\vec{J}(\vec{r}, t)$ over all of space since this integral should be $\propto I(t) 2\pi a$]

c) From Eq. (9b) it follows [$\vec{m}(t) = \vec{m}_0 e^{-i\omega t}$]

$$\begin{aligned} \vec{m}_0 &= \frac{1}{2} \int d^3r \vec{r} \times \vec{J}(\vec{r} | \omega) \\ &= \frac{\hat{z}}{2} \frac{I_0}{a} \int r \delta(r-a) \delta(\theta-\pi/2) r^2 \sin\theta dr d\theta d\phi \\ &= \frac{\hat{z}}{2} \frac{I_0}{a} 2\pi a^3 \\ &= \hat{z} I_0 \pi a^2 \end{aligned} \quad (3-5)$$

so that

$$\vec{m}(t) = \hat{z} I_0 \pi a^2 e^{-i\omega t} \quad (3-6)$$

which is the time-dependent dipole moment.

d) With this expression for \vec{m}_0 we get for

$$\begin{aligned}\vec{F}(\theta, \phi) &= -ik \vec{m}_0 \times \hat{r} \\ &= -ik m_0 \sin\theta \\ &= -ik I_0 \pi a^2 \sin\theta\end{aligned}\quad (3-7)$$

so that the vector potential becomes in the far field regions

$$\begin{aligned}\vec{A}(\vec{r}, t) &= \frac{\mu_0}{4\pi} \frac{e^{ikr-i\omega t}}{r} (-ik I_0 \pi a^2 \sin\theta) \hat{\phi} \\ &= -\frac{ik \mu_0 I_0 a^2 \sin\theta}{4} \frac{e^{ikr-i\omega t}}{r} \hat{\phi}\end{aligned}\quad (3-8)$$

which has the behavior we should show.

d) Radiation fields are electromagnetic (E- or H-fields) which decay like $1/r$ in the far field region.

Since the fields can be calculated from $\vec{A}(\vec{r}, t)$, and we only here are interested in radiation fields, it is sufficient to use $\lim_{kr \rightarrow \infty} \vec{A}(\vec{r}, t)$ for this calculation.

The magnetic field is given by

$$\vec{H}(\vec{r}, t) = \frac{1}{\mu_0} \nabla \times \vec{A}(\vec{r}, t)$$

Using Eq (3-8) one gets in the far field

$$\begin{aligned} \vec{H}(\vec{r}, t) &= \frac{-ikI_0a^2}{4} \nabla \times \left(\sin\theta \frac{e^{ikr-i\omega t}}{r} \hat{\phi} \right) \\ &= \frac{-ikI_0a^2}{4} \left[-\partial_r \left\{ \sin\theta \frac{e^{ikr-i\omega t}}{r} \right\} \hat{\theta} \right. \\ &\quad \left. + O(1/r^2) \right] \\ &= \frac{ikI_0a^2 \sin\theta (ik \cdot r - 1) e^{ikr-i\omega t}}{4 r^2} \hat{\theta} \\ &\quad + O(1/r^2) \\ &= -\frac{k^2 I_0 a^2 \sin\theta}{4} \frac{e^{ikr-i\omega t}}{r} \hat{\theta} + O(1/r^2) \end{aligned} \tag{3-9}$$

Here the radiation H-field is the first term without the $O(1/r^2)$. Note from (3-9) that this radiation field also can be written as:

$$\vec{H}(\vec{r}, t) = \frac{1}{\mu_0} \nabla \times \vec{A}(\vec{r}, t) = \frac{1}{\mu_0} i\vec{k} \times \vec{A}(\vec{r}, t)$$

with $\vec{k} = k\hat{r}$ and using $\hat{r} \times \hat{\phi} = -\hat{\theta}$.

The latter result follows from observing that $\vec{A}(\vec{r}, t)$ has the form of a spherical wave with an angular dependent amplitude, that locally in the far field behaves like a plane wave.

The electric field (in the far field) follows from Eq. (3-9) by the use of ampers law:

$$-i\omega\epsilon_0 \vec{E}(\vec{r}, t) = \nabla \times \vec{H}$$

$$\begin{aligned} \vec{E}(\vec{r}, t) &= \frac{i}{\omega\epsilon_0} \nabla \times \vec{H} \\ &= \frac{i}{\omega\epsilon_0} ik \vec{r} \times \vec{H} \\ &= -\frac{1}{\omega\epsilon_0} \vec{k} \times \vec{H}, \quad \vec{k} = k\hat{r} \\ &= \frac{k^3 I_0 a^2}{4\omega\epsilon_0} \sin\theta \frac{e^{ikr-i\omega t}}{r} \underbrace{\hat{r} \times \hat{\theta}}_{\hat{\phi}} \end{aligned} \quad (3-10)$$

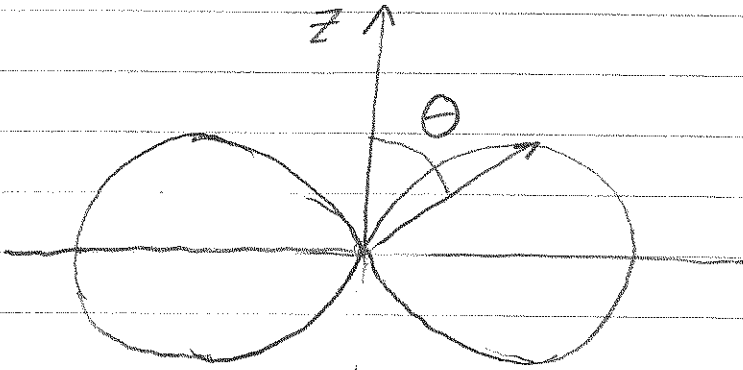
This result one could also have obtained by the direct method used to calculate \vec{H} .

e) The time average Poynting vector (in the far field) becomes

$$\begin{aligned} \langle \vec{S} \rangle &= \frac{1}{2} \vec{E} \times \vec{H}^* \\ &= +\frac{1}{2} \frac{1}{\omega\epsilon_0} \vec{H}^* \times (\vec{k} \times \vec{H}) \\ &= \frac{1}{2} \frac{1}{\omega\epsilon_0} \left[\vec{k} |\vec{H}|^2 - \vec{H} (\vec{H}^* \cdot \vec{k}) \right] \\ &= \frac{1}{2} \frac{k}{\omega\epsilon_0} \frac{k^4 I_0^2 a^4}{16} \frac{\sin^2\theta}{r^2} \hat{r} \\ &= \frac{\mu_0 \omega^4 I_0^2 a^4}{32 c^3} \frac{\sin^2\theta}{r^2} \hat{r} \end{aligned} \quad (3-11)$$

-3.7-

$$\begin{aligned}\frac{dP}{d\Omega} &= r^2 \hat{r} \cdot \langle \vec{S} \rangle \\ &= \frac{\mu_0 \omega^4 I_0^2 a^4}{32 c^3} \sin^2 \theta\end{aligned}\quad (3-12)$$



f] The total power radiated by the system is :

$$\begin{aligned}P &= \int d\Omega \frac{dP}{d\Omega} \\ &= \frac{\mu_0 \omega^4 I_0^2 a^4}{32 c^3} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \sin^2 \theta \\ &= \frac{\mu_0 \omega^4 I_0^2 a^4}{32 c^3} 2\pi \int_0^\pi \sin^3 \theta d\theta \\ &= \frac{\mu_0 \omega^4 I_0^2 a^4}{32 c^3} 2\pi \underbrace{\int_{-1}^1 (1-x^2) dx}_{4/3}, \quad x = \cos \theta \\ &= \frac{\pi \mu_0 \omega^4 I_0^2 a^4}{12 c^3}\end{aligned}\quad (3-13)$$

g) Since $P \propto I_0^2 a^4$ it follows if $I_0 \rightarrow 2I_0$ then $a \rightarrow a/\sqrt{2}$ for the product $I_0^2 a^4$ to remain unchanged.

Therefore, if I_0 is doubled, the radius has to be reduced by a factor $\sqrt{2}$ in order for the radiated power to remain unchanged.

Since $P \propto m_0^2$, the amplitude of the dipole moment, it follows that if m_0 is unchanged, so is P . Moreover, since $m_0 = I_0 A$, where $A = \pi a^2$, our conclusion made above is reasonable.