TFY4240

Exam 2014 Dec 3.

Problem 1

a) The charge density is:

$$\rho(\vec{r}) = q\,\delta(\vec{r} - \hat{x}_3 d)$$

The eq. to be satisfied are

* above the surface:

$$\nabla \vec{D} = \rho(\vec{r}) \;; \quad \nabla\cdot\vec{E} = \rho(\vec{r})/\varepsilon_0\varepsilon_1$$
$$\nabla\times\vec{E} = 0$$

* below the surface

$$\nabla \vec{D} = 0 \;; \quad \nabla\cdot\vec{E} = 0$$
$$\nabla\times\vec{E} = 0$$

b) Since $\vec{E} = -\nabla V(\vec{r})$ under electrostatic conditions, it follows that

$$\nabla^2 V(\vec{r}) = \begin{cases} -\dfrac{\rho(\vec{r})}{\varepsilon_0\varepsilon_1} & x_3 > 0 \\ 0 & x_3 < 0 \end{cases}$$

The eq. $\nabla\times\vec{E}$ is always satisfied.

Boundary conditions

i] $X_3 = 0$

$$V_+(\vec{r})\Big|_{\vec{r}=\vec{r}_{\parallel}} = V_-(\vec{r})\Big|_{\vec{r}=\vec{r}_{\parallel}}$$

$$\varepsilon_1 \partial_3 V_+(\vec{r})\Big|_{\vec{r}=\vec{r}_{\parallel}} = \varepsilon_2 \partial_3 V_-(\vec{r})\Big|_{\vec{r}=\vec{r}_{\parallel}}$$

ii] $r \to \pm\infty$

$$V(\vec{r}) \to 0$$

c] Image method:

Since the metal is grounded we choose:

$$V_-(\vec{r}) = 0$$

and the method of images gives for the potential in region 1:

$$V_+(\vec{r}) = \frac{1}{4\pi\varepsilon_0\varepsilon_1}\left[\frac{q}{|\vec{r}-d\hat{x}_3|} - \frac{q}{|\vec{r}+d\hat{x}_3|}\right]$$

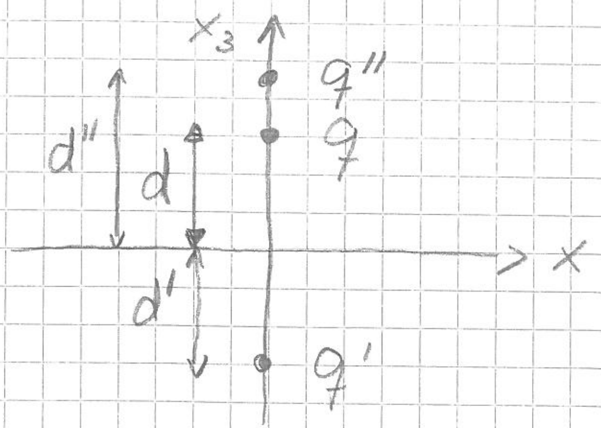
where

$$|\vec{r} \pm d\hat{x}_3| = \sqrt{x_{\parallel}^2 + (X_3 \pm d)^2}$$

One may check explicitly that the BC are satisfied at $X_3 = 0$

d) In this dielectric case the potential at $x_3 = 0$ is no longer zero. Hence the situation is somewhat more complicated:

For symmetry reasons any image charges are expected to be placed on the x_3-axis.



$$V_+(\vec{r}) = \frac{1}{4\pi\varepsilon_0\varepsilon_1}\left[\frac{q}{|\vec{r}-d\hat{x}_3|} + \frac{q'}{|\vec{r}+d'\hat{x}_3|}\right]$$

where

$$|\vec{r}-d\hat{x}_3| = \sqrt{x_{\parallel}^2 + (x_3-d)^2} \qquad x_{\parallel}^2 = x_1^2 + x_2^2$$

$$|\vec{r}+d'\hat{x}_3| = \sqrt{x_{\parallel}^2 + (x_3+d')^2}$$

$$V_-(\vec{r}) = \frac{1}{4\pi\varepsilon_0\varepsilon_2}\,\frac{q''}{|\vec{r}-d''\hat{x}_3|}$$

$$|\vec{r}-d''\hat{x}_3| = \sqrt{x_{\parallel}^2 + (x_3-d'')^2}$$

e) The BC at infinity is satisfied. However, we have to impose the BC at $x_3 = 0$

i) $$\frac{1}{\varepsilon_1}\left[\frac{q}{\sqrt{x_{\parallel}^2 + d^2}} + \frac{q'}{\sqrt{x_{\parallel}^2 + d'^2}}\right]$$

$$= \frac{1}{\varepsilon_2} \frac{q''}{\sqrt{x_{\parallel}^2 + d''^2}} \qquad (1.1a)$$

ii) $$\partial_3 \left[x_{\parallel}^2 + (x_3 - a)^2\right]^{-1/2}$$

$$= -\frac{1}{2}\left[x_{\parallel}^2 + (x_3 - a)^2\right]^{-3/2} 2(x_3 - a)$$

$$= \frac{x_3 - a}{\left[x_{\parallel}^2 + (x_3 - a)^2\right]^{3/2}}$$

$$\frac{q(-d)}{\left[x_{\parallel}^2 + d^2\right]^{3/2}} + \frac{q' d'}{\left[x_{\parallel}^2 + d'^2\right]^{3/2}}$$

$$= \frac{q''(-d'')}{\left[x_{\parallel}^2 + d''^2\right]^{3/2}} \qquad (1.1b)$$

Eq (1.1) should hold for any $x_{\parallel}$, in part. for $x_{\parallel} = 0$ and $x_{\parallel} \gg d, d'$ and d'' (or $x_{\parallel} \to \infty$)

$$\frac{q}{\varepsilon_1 d} + \frac{q'}{\varepsilon_1 d'} = \frac{q''}{\varepsilon_2 d''} \qquad (1.2a)$$

$$\frac{-qd}{d^3} + \frac{q'd'}{d'^3} = -\frac{q''d''}{d''^3}$$

$$-\frac{q}{d^2} + \frac{q'}{d'^2} = -\frac{q''}{d''^2} \qquad (1.2b)$$

Equation (1.2) represents two eqs. for four unknown. Hence we have an under-determined system.

We use this to try to find a solution when

$$d = d' = d''!$$

This choice is motivated by what one has in the metallic case.

Now, Eq. (1.2) becomes

$$\frac{q}{\varepsilon_1} + \frac{q'}{\varepsilon_1} = \frac{q''}{\varepsilon_2} \qquad (1.3a)$$

$$-q + q' = -q'' \qquad (1.3b)$$

or

$$\begin{bmatrix} \frac{1}{\varepsilon_1} & -\frac{1}{\varepsilon_2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} q' \\ q'' \end{bmatrix} = \begin{bmatrix} -\frac{1}{\varepsilon_1} \\ 1 \end{bmatrix} q$$

The solution is :

$$q' = -\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} q$$

$$q'' = \frac{2\varepsilon_2}{\varepsilon_2 + \varepsilon_1} q$$

and the potential in the two regions is determined.

f) If $|\varepsilon_2| \gg \varepsilon_1$ one finds

$$q' = -\frac{1 - \frac{\varepsilon_1}{\varepsilon_2}}{1 + \frac{\varepsilon_1}{\varepsilon_2}} q \simeq -q$$

$$q'' = \frac{2}{1 + \frac{\varepsilon_1}{\varepsilon_2}} q \simeq 2q$$

Hence, the potential $V_-(\vec{r})$ will become small and $V_+(\vec{r})$ appraches what we have for the potential above a grounded metal plate.

In the extreme case $|\varepsilon_2| \to \infty$ V_- becomes zero.

Problem 2

a) Maxwells eqs. with sources

$$\nabla\cdot\vec{D} = \rho \qquad (2.1a)$$

$$\nabla\cdot\vec{B} = 0 \qquad (\text{" } b)$$

$$\nabla\times\vec{E} = -\partial_t\vec{B} \qquad (\text{" } c)$$

$$\nabla\times\vec{H} = \vec{J}+\partial_t\vec{D} \qquad (\text{" } d)$$

$(2.1d)\times\vec{B} - \vec{D}\times(2.1c)$ gives:

$$(\nabla\times\vec{H})\times\vec{B} - \vec{D}(\nabla\times\vec{E}) = \vec{J}\times\vec{B}+(\partial_t\vec{D})\times\vec{B}+\vec{D}\times\partial_t\vec{B}$$

$$\partial_t(\vec{D}\times\vec{B}) + \vec{J}\times\vec{B}$$
$$+\ \vec{D}\times(\nabla\times\vec{E}) + \vec{B}\times(\nabla\times\vec{H}) - \underbrace{\left[\vec{E}(\nabla\cdot\vec{D}) - \rho\vec{E}\right]}_{0}$$
$$-\ \vec{H}(\nabla\cdot\vec{B}) \quad = 0$$

$$\partial_t(\vec{D}\times\vec{B}) + \rho\vec{E} + \vec{J}\times\vec{B}$$
$$+\left\{\vec{D}\times(\nabla\times\vec{E}) - \vec{E}(\nabla\cdot\vec{D}) + \vec{B}\times(\nabla\times\vec{H}) - \vec{H}(\nabla\cdot\vec{B})\right\} = 0$$

Hence, Eq (1a) of the exam set is derived and we find

$$\vec{f} = \rho\vec{E}+\vec{J}\times\vec{B}$$

The vector $\vec{f}$ describes the force density acting at position $\vec{r}$ at time t.

b] Lets look at the i'th component

$$[\vec{D}\times(\nabla\times\vec{E}) - \vec{E}(\nabla\cdot\vec{D})]_i$$

$$= \varepsilon_{ijk} D_j (\nabla\times\vec{E})_k - E_i(\nabla\cdot\vec{D})$$

$$= \varepsilon_{ijk} D_j \varepsilon_{kmn} \partial_m E_n - E_i(\nabla\cdot\vec{D})$$

$$= \varepsilon_{kij}\varepsilon_{kmn} D_j \partial_m E_n - E_i(\nabla\cdot\vec{D})$$

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) D_j \partial_m E_n - E_i(\nabla\cdot\vec{D})$$

$$= \underbrace{D_j \partial_i E_j}_{\frac{1}{2}\partial_i(\vec{E}\cdot\vec{D})} \underbrace{- D_j\partial_j E_i - E_i \partial_j D_j}_{-\partial_j(E_i D_j)}$$

$$= -\partial_j(E_i D_j) + \frac{1}{2}\partial_i(\vec{E}\cdot\vec{D})$$

$$= \partial_j\left[E_i D_j + \delta_{ij}\frac{\vec{E}\cdot\vec{D}}{2}\right]$$

Similar expression for the magnetic term $(\vec{E}\rightarrow\vec{H},\ \vec{D}\rightarrow\vec{B})$

Therefore

$$\{\ \ \}_j = \partial_i T_{ij}$$

with

$$T_{ij} = D_i E_j + B_i H_j + \delta_{ij}\left(\frac{\vec{D}\cdot\vec{E}}{2} + \frac{\vec{B}\cdot\vec{H}}{2}\right)$$

Since $\vec{D} = \varepsilon_0 \vec{E}$ and $\vec{B} = \mu_0 \vec{H}$ this tensor is indeed symmetric

c) Integrating Eq (2a) of the exam set over a small volum V gives

$$\int_V d^3r \left(\partial_t \vec{g} + \vec{f}\right) + \int_V d^3r \, \nabla \cdot \overleftrightarrow{T} = 0$$

$$\frac{d}{dt}\vec{G} + \vec{F} + \int_{\partial V} d\vec{S} \cdot \overleftrightarrow{T} = 0$$

This means that the time rate of change of the total momentum inside V pluss the total force acting on the volum

ADD

Problem 3

a) Dipole moment

$$\vec{p}(t) = \int d^3r \; \vec{r}\, \rho(\vec{r},t)$$

$$= e^{-i\omega t} \int d^3r \; \vec{r} \; Q[\delta(x_3-a) - \delta(x_3+a)]\, \delta(x_1)\delta(x_2)$$

$$= e^{-i\omega t} Q \hat{x}_3 (a-(-a))$$

$$= 2aQ e^{-i\omega t} \; \hat{x}_3 \qquad (3.1)$$

The current density can be obtained from the continuity eq.

$$\nabla \vec{J} = -\partial_t \rho(\vec{r},t) = -i\omega\, \rho(\vec{r},t)$$

Since $\vec{J} \propto \hat{x}_3$ one gets by a simple integration

$$\vec{J}(\vec{r}|\omega) = -i\omega \int dx_3 \; \rho(\vec{r}|\omega)\; \hat{x}_3$$

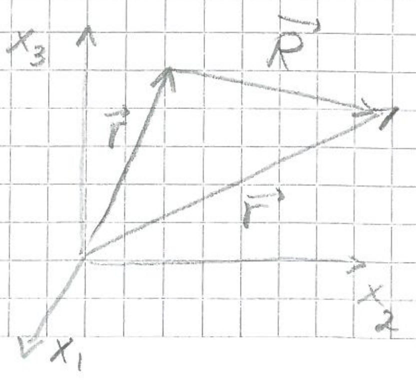
$$= -i\omega Q\, \delta(x_1)\delta(x_2)\, \Theta(a-|x_3|)\, \hat{x}_3 \quad (3.2)$$

b) $\vec{r}$: observation point

$\vec{r}'$: source point

R : source-observer seperation : $\vec{R} = \vec{r} - \vec{r}'$

t_r : retarded time : $t_r = t - R/c$



The radiation zone is the region where $kR \gg 1$ (and $ka \ll 1$). In this region radiation fields will dominate if present.

$$R = |\vec{r} - \vec{r}\,'|$$
$$= r\left(1 - 2\frac{\hat{r}\cdot\vec{r}\,'}{r} + \frac{r'^2}{r^2}\right)^{1/2}$$
$$\simeq r - \hat{r}\cdot\vec{r}\,'$$

$$\frac{1}{R} \simeq \frac{1}{r}\left(1 + \frac{\hat{r}\cdot\vec{r}\,'}{r} + \dots\right)$$

In the radiation zone the vector potential becomes:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\vec{J}(\vec{r}\,'|\omega)}{R}\,e^{-i\omega(t-\frac{R}{c})}$$
$$\simeq \frac{\mu_0}{4\pi}\,\frac{e^{-i\omega t}}{r}\int d^3r'\,\vec{J}(\vec{r}\,'|\omega)\,e^{ikr}\,e^{-ik\hat{r}\cdot\vec{r}\,'}$$
$$= \frac{\mu_0}{4\pi}\,\frac{e^{ikr-i\omega t}}{r}\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\int d^3r'\,\vec{J}(\vec{r}\,'|\omega)\,(\hat{r}\cdot\vec{r}\,')^n$$

c) The lowest order contribution is:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{e^{+ikr-i\omega t}}{r} \int d^3r' \, \vec{J}(\vec{r}'|\omega)$$

$$= -i\omega Q \frac{\mu_0}{4\pi} \frac{\exp(ikr-i\omega t)}{r} \int_{-a}^{a} dx_3' \, 1 \, \hat{x}_3$$

$$= -i\omega \frac{\mu_0}{4\pi} 2aQ \frac{\exp(ikr-i\omega t)}{r} \hat{x}_3$$

$$= \frac{\mu_0}{4\pi} \dot{\vec{p}}(t) \frac{e^{ikr}}{r}$$

The magnetic field (in the radiation zone)

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0} \nabla \times \vec{A}(\vec{r},t)$$

$$H_i = \frac{1}{\mu_0} \varepsilon_{ijk} \partial_j \left(\frac{\mu_0}{4\pi} \dot{p}_k \frac{e^{ikr}}{r} \right)$$

$$= \frac{1}{4\pi} \varepsilon_{ijk} \dot{p}_k \partial_j \left(\frac{\exp(ikr)}{r} \right)$$

$$= \frac{1}{4\pi} \varepsilon_{ijk} \dot{p}_k \, ik \frac{e^{ikr}}{r} \partial_j r + O(1/r^2)$$

$$\partial_j r = \partial_j \sqrt{x_i x_i} = (\hat{r})_j$$

$$= \frac{1}{4\pi} \varepsilon_{ijk} \hat{r}_j \dot{p}_k \frac{e^{ikr}}{r} (ik)$$

$$= \frac{ik}{4\pi} (\hat{r} \times \dot{\vec{p}})_i \frac{e^{ikr}}{r}$$

$$= \frac{ck^2}{4\pi} (\hat{r} \times \vec{p}(t))_i \frac{e^{ikr}}{r}$$

Hence in the radiation zone one gets

$$\vec{H}(\vec{r},t) = \frac{ck^2}{4\pi} (\hat{r} \times \vec{p}(t)) \frac{e^{ikr}}{r}$$

The electric field becomes (from Ampere's law)

$$\vec{E}(\vec{r},t) = \frac{1}{-i\omega\varepsilon_0} \nabla \times \vec{H}$$

$$= \frac{1}{-i\omega\varepsilon_0} \, i\vec{k} \times \vec{H} \quad + O(1/r^2)$$

$$= \frac{1}{\omega\varepsilon_0} \frac{ck^2}{4\pi} k \, \hat{r} \times (\hat{r} \times \vec{p}(t)) \frac{e^{ikr}}{r}$$

$$= \frac{k^2}{4\pi\varepsilon_0} \hat{r} \times [\hat{r} \times \vec{p}(t)] \frac{e^{ikr}}{r}$$

The time averaged Poynting's vector:

$$\langle \vec{S} \rangle = \frac{1}{2} \vec{E} \times \vec{H}^*$$

$$= \frac{-1}{2} \frac{1}{\omega\varepsilon_0} (\vec{k} \times \vec{H}) \times \vec{H}^*$$

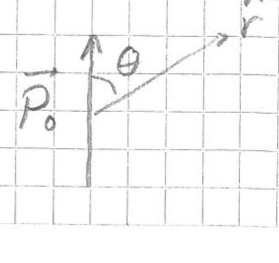
$$= \frac{+1}{2} \frac{1}{\omega\varepsilon_0} \vec{H}^* \times (\vec{k} \times \vec{H})$$

$$= \frac{1}{2} \frac{1}{\omega\varepsilon_0} \left[\vec{k} |\vec{H}|^2 - \vec{H} \underbrace{(\vec{k} \cdot \vec{H}^*)}_{0} \right]$$

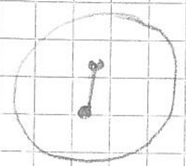
$$= \frac{1}{2} \frac{k}{\omega\varepsilon_0} \frac{c^2 k^4}{(4\pi)^2} \frac{|\hat{r} \times \vec{p}_0|^2}{r^2} \hat{r}, \quad \vec{p}_0 = 2aQ\hat{x}_3, \quad \vec{k} = k\hat{r}$$

$$= \frac{1}{32\pi^2\varepsilon_0} \frac{\omega^4}{c^3} \frac{|\hat{r} \times \vec{p}_0|^2}{r^2} \hat{r}$$

$$= \frac{1}{8\pi^2\varepsilon_0} \frac{\omega^4}{c^3} a^2 Q^2 \frac{\sin^2\theta}{r^2} \hat{r}$$



d) The time-averaged power crossing a surfase A is :



$$\bar{P} = \int \langle \vec{S} \rangle \, d\vec{A}$$

$$= \int \langle \vec{S} \rangle \cdot \hat{r} \; r^2 d\Omega$$

$$\equiv \int \left\langle \frac{dP}{d\Omega} \right\rangle d\Omega$$

$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle = \langle \vec{S} \rangle \cdot \hat{r} \; r^2$$

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{8\pi^2 \varepsilon_0} \, \frac{\omega^4}{c^3} a^2 Q^2 \sin^2\theta$$

e) In the Lorentz gauge one has

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{J}(\vec{r}', t_r)}{R} \qquad R = |\vec{r} - \vec{r}'|$$

$$= \frac{\mu_0}{4\pi} (-i\omega Q) \int d^3r' \, \delta(x_1')\delta(x_2')\,\theta(a - |x_3'|)\,\hat{x}_3 \; e^{-i\omega(t - |\vec{r}-\vec{r}'|/c)} / |\vec{r}-\vec{r}'|$$

$$= -i \frac{\mu_0 \omega Q}{4\pi} e^{-i\omega t} \int_{-a}^{a} dx_3' \, \frac{e^{ik|\vec{r} - x_3'\hat{x}_3|}}{|\vec{r} - x_3'\hat{x}_3|} \, \hat{x}_3$$

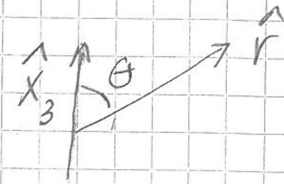
f] Under the assumption $r \gg a$ it follows that (since $|x_3'| < a$)

$$|\vec{r} - x_3'\hat{x}_3| = \left[r^2 - 2\vec{r}\cdot\hat{x}_3 x_3' + x_3'^2\right]^{1/2}$$

$$= r\left[1 - 2\frac{x_3'}{r}\cos\theta + \left(\frac{x_3'}{r}\right)^2\right]^{1/2}$$

$$\simeq r\left(1 - \frac{x_3'}{r}\cos\theta + \cdots\right)$$

$$= r - x_3'\cos\theta + \cdots$$



However in the expression

$$\frac{1}{|\vec{r} - x_3'\hat{x}_3|} \simeq \frac{1}{r}.$$

This gives :

$$\vec{A}(\vec{r},t) = -i\frac{\mu_0 \omega Q}{4\pi} e^{-i\omega t} \times \int_{-a}^{a} dx_3' \, \frac{e^{ikr - ikx_3'\cos\theta}}{r}\,\hat{x}_3$$

$$= -i\frac{\mu_0 \omega Q}{4\pi} \frac{e^{ikr - i\omega t}}{r} \underbrace{\int_{-a}^{a} dx_3' \, e^{-ikx_3'\cos\theta}}_{\frac{2}{k\cos\theta}\sin(ka\cos\theta)}\,\hat{x}_3$$

$$=$$

$$= -i\frac{\mu_0 \omega Q}{4\pi}\frac{2}{k}\frac{e^{ikr-i\omega t}}{r}\frac{\sin(ka\cos\theta)}{\cos\theta}\hat{x}_3$$

$$= -i\frac{\mu_0 c Q}{2\pi}\frac{e^{ikr-i\omega t}}{r}\frac{\sin(ka\cos\theta)}{\cos\theta}\hat{x}_3$$

g) In sherical coordinates

$$\hat{x}_3 = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

and since $\vec{A}$ is independent of ϕ it follows that:

$$\vec{H} = \frac{1}{\mu_0}\nabla\times\vec{A}$$

$$\simeq \frac{1}{\mu_0} i\vec{k}\times\vec{A} \quad \text{(in the radiation zone)}$$

The electric field follows from Amperes law.

$$\nabla\times\vec{H} = -i\omega\vec{D} = -i\omega\varepsilon_0\vec{E}$$

$$\Rightarrow \vec{E} = \frac{1}{-i\omega\varepsilon_0}\nabla\times\vec{H}$$

$$\simeq \frac{1}{-i\omega\varepsilon_0} i\vec{k}\times\vec{H} \quad \text{(radiation)}$$

$$= -\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\hat{k}\times\vec{H}$$

$$\langle \vec{S} \rangle = \frac{1}{2} \vec{E} \times \vec{H}^*$$

$$= \frac{1}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}} \quad \vec{H}^* \times (\hat{k} \times \vec{H})$$

$$= \frac{1}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}} \left[\hat{k} |\vec{H}|^2 - \vec{H} \underbrace{(\vec{H}^* \cdot \hat{k})}_{0} \right]$$

$$= \frac{1}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}} |\vec{H}|^2 \hat{k}$$

$$= \frac{1}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{k^2}{\mu_0^2} |\hat{k} \times \vec{A}|^2 \hat{r} \quad , \quad \hat{k} = \hat{r}$$

$$= \frac{1}{2} \frac{\omega^2}{c^2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{\mu_0^2} \quad |\vec{A}|^2 \sin^2\theta \ \hat{r}$$

$$= \frac{\omega^2 Q^2}{8\pi^2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{r^2} \frac{\sin^2(ka\cos\theta)}{\cos^2\theta} \sin^2\theta \ \hat{r}$$

h)

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega^2 Q^2}{8\pi^2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\sin^2(ka\cos\theta)}{\cos^2\theta} \sin^2\theta$$

i] When $ka \ll 1$ it follows that

$$\sin(ka\cos\theta) \simeq ka\cos\theta$$

so that

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega^2 Q^2}{8\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^2 a^2 \sin^2\theta$$

Rewriting the prefactor

$$\omega^2 Q^2 \sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^2 a^2$$

$$= \omega^2 Q^2 \sqrt{\frac{\mu_0}{\varepsilon_0}}\, \frac{\omega^2}{c^2} a^2$$

$$= \frac{\omega^4 Q^2 a^2}{c^3}$$

gives

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega^4 a^2 Q^2}{8\pi^2 \varepsilon_0 c^3} \sin^2\theta$$

This is the same expression we obtained in 3d]