TFY4240 Electromagnetic theory: Solution to exam, Dec 2015

Problem 1

(a) The problem is to find the potential V in some region Ω of a physical system, given (i) the charge density in Ω and (ii) V on the boundary of Ω. The method consists of constructing an alternative system where (i) and (ii) are unchanged, and solving the problem for the alternative system instead (because a uniqueness theorem guarantees that the solution for V in Ω will be the same in both systems). The alternative system contains image charges in the region outside of Ω, hence the name of the method.

(b) The potential is, for $|\boldsymbol{r}| \geq R$,

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\boldsymbol{r} - a\hat{\boldsymbol{z}}|} + \frac{q'}{|\boldsymbol{r} - b\hat{\boldsymbol{z}}|}\right]. \tag{1}$$

The boundary condition (BC) is $V(\boldsymbol{r}) = 0$ for all $\boldsymbol{r}$ with $|\boldsymbol{r}| = R$. To find the two unknowns q' and b, we can consider the BC for two special cases, say $\boldsymbol{r} = \pm R\hat{\boldsymbol{z}}$. This gives

$$\text{for } \boldsymbol{r} = +R\hat{\boldsymbol{z}}: \qquad \frac{q}{|R-a|} + \frac{q'}{|R-b|} = 0 \quad \Rightarrow \quad q' = -\frac{R-b}{a-R}q, \tag{2}$$

$$\text{for } \boldsymbol{r} = -R\hat{\boldsymbol{z}}: \qquad \frac{q}{|-R-a|} + \frac{q'}{|-R-b|} = 0 \quad \Rightarrow \quad q' = -\frac{R+b}{R+a}q. \tag{3}$$

where we used that $a > R$ and $b < R$. Equating the two expressions for q' gives

$$(R+a)(R-b) = (R+b)(a-R) \quad \Rightarrow \quad 2R^2 = 2ab \quad \Rightarrow \quad b = \frac{R^2}{a}. \tag{4}$$

Inserting this result for b into one of the equations for q', say Eq. (3), gives

$$q' = -\frac{R + R^2/a}{R+a}q = -\frac{R}{a}\cdot\frac{1+R/a}{R/a+1}q = -\frac{R}{a}q. \tag{5}$$

We should now check whether this solution for q' and b also satisfies the BC's for the general case $|\boldsymbol{r}| = R$ (after all, while getting a solution to our set of two linear equations (2)-(3) in two unknowns was mathematically guaranteed, it is a priori not obvious that we would get the *same* solution regardless of which two special points on the spherical surface we selected). To this end, let us write

$$|\boldsymbol{r} - c\hat{\boldsymbol{z}}| = \sqrt{(\boldsymbol{r} - c\hat{\boldsymbol{z}})\cdot(\boldsymbol{r} - c\hat{\boldsymbol{z}})} = \sqrt{r^2 - 2c\boldsymbol{r}\cdot\hat{\boldsymbol{z}} + c^2} = \sqrt{r^2 - 2rc\cos\theta + c^2}. \tag{6}$$

Using this result, the second term inside the square brackets in (1) becomes, for $|\boldsymbol{r}| = R$,

$$\frac{-qR/a}{\sqrt{R^2 - 2R\cdot(R^2/a)\cos\theta + (R^2/a)^2}} = -\frac{q}{\sqrt{R^2 - 2Ra\cos\theta + a^2}}, \tag{7}$$

which is the negative of the first term, confirming the BC for an arbitrary point on the spherical surface.

(c) The surface charge density σ is given by

$$\sigma = -\epsilon_0\left[\left.\frac{\partial V}{\partial n}\right|_{\text{outside}} - \left.\frac{\partial V}{\partial n}\right|_{\text{inside}}\right] = -\epsilon_0\left.\frac{\partial V}{\partial n}\right|_{\text{outside}}. \tag{8}$$

Here "outside" ("inside") refer to evaluating the derivatives just outside (inside) the spherical surface. The "inside" term vanishes since the sphere is a conductor and thus an equipotential in electrostatics. Since the surface normal has the same direction as $\hat{\boldsymbol{r}}$, it follows that $\partial/\partial n = \partial/\partial r$. Thus

$$\begin{aligned}
\sigma &= -\frac{1}{4\pi}\frac{\partial}{\partial r}\left[\frac{q}{\sqrt{r^2 - 2ra\cos\theta + a^2}} + \frac{q'}{\sqrt{r^2 - 2rb\cos\theta + b^2}}\right]\Bigg|_{r=R} \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q(R - a\cos\theta)}{(R^2 - 2Ra\cos\theta + a^2)^{3/2}} + \frac{q'(R - b\cos\theta)}{(R^2 - 2Rb\cos\theta + b^2)^{3/2}}\right] \\
&= \frac{q}{4\pi}\frac{R^2 - a^2}{R(R^2 + a^2 - 2Ra\cos\theta)^{3/2}}.
\end{aligned} \tag{9}$$

As is reasonable, this expression for σ has the opposite sign of q and its magnitude decreases with θ. Also, its dimension is [charge]/[length]^2, as it should be (it is good to make such checks).

The total charge of the entire system (point charge + sphere) is $q + Q \equiv Q_{\rm tot}$. Here, $Q_{\rm tot}$ is also the charge appearing in the monopole term $Q_{\rm tot}/4\pi\epsilon_0 r$ in the multipole expansion of the potential. From (1) one can see that the monopole term is $(q+q')/4\pi\epsilon_0 r$, so $Q_{\rm tot} = q + q'$, giving

$$Q = Q_{\rm tot} - q = (q + q') - q = q'. \tag{10}$$

Alternatively, Q can be found by integrating the surface charge density σ over the spherical surface:

$$Q = \int \sigma da = R^2 \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \, \sigma = \frac{qR(R^2 - a^2)}{2} \int_{-1}^{1} \frac{dx}{(R^2 + a^2 - 2Rax)^{3/2}} \tag{11}$$

(here the φ-integral just gave a factor 2π and we changed integration variables from θ to $x = \cos\theta$). The integral is $\int_{-1}^{1} dx \, (C + Dx)^{-3/2}$ with constants $C = R^2 - a^2$ and $D = -2Ra$. Changing integration variable to $u = C + Dx$, the integral becomes

$$\frac{1}{D} \int_{C-D}^{C+D} du \, u^{-3/2} = \frac{1}{D} \cdot \frac{1}{-3/2 + 1} u^{-3/2+1} \Bigg|_{C-D}^{C+D} = -\frac{2}{D} \left[\frac{1}{\sqrt{C+D}} - \frac{1}{\sqrt{C-D}} \right]. \tag{12}$$

Using $\sqrt{C \pm D} = \sqrt{R^2 + a^2 \mp 2Ra} = \sqrt{(R \mp a)^2} = a \mp R$, we get

$$Q = \frac{qR(R^2 - a^2)}{2} \cdot \frac{(-2)}{(-2Ra)} \underbrace{\left[\frac{1}{a - R} - \frac{1}{a + R} \right]}_{2R/(a^2 - R^2)} = -q\frac{R}{a} = q'. \tag{13}$$

(e) Call the second image charge q''. Since q and q' together make $V = 0$ at $r = R$, the job of q'' is to raise the potential from 0 to V_0 at $r = R$. Since all points with $|\boldsymbol{r}| = R$ should be raised by the same value V_0, q'' must be positioned equally far away from all these points, and therefore it must be placed at the origin $r = 0$. Its potential at $r = R$ is therefore $q''/4\pi\epsilon_0 R$. This should equal V_0, so $q'' = 4\pi\epsilon_0 R V_0$. The potential outside the sphere is $V(\boldsymbol{r}) = (4\pi\epsilon_0)^{-1}(q/|\boldsymbol{r} - a\hat{\boldsymbol{z}}| + q'/|\boldsymbol{r} - b\hat{\boldsymbol{z}}| + q''/|\boldsymbol{r}|)$.

Problem 2

(a) The field produced by wire i ($i = 1, 2$) will be "circumferential" with a direction given by a right-hand rule. The magnitude can be found by applying Stokes' theorem ("the curl theorem" in the formula set) to Ampere's law (note that since there is no changing electric field, there is no displacement current, so the Ampere-Maxwell law reduces to Ampere's law)

$$\nabla \times \boldsymbol{B}_i = \mu_0 \boldsymbol{J}_i \quad \Rightarrow \quad \oint_C \boldsymbol{B}_i \cdot d\boldsymbol{s} = \mu_0 I \tag{14}$$

where C refers to an "Amperean" loop of circular shape with radius r centered on wire i and I is the current through wire i. Since $\boldsymbol{B}_i$ points along $d\boldsymbol{s}$ and has constant magnitude on the loop, we get $B_i \cdot 2\pi r = \mu_0 I$, so $B_i = \mu_0 I/2\pi r$. At wire 1, $\boldsymbol{B}_2 = \hat{\boldsymbol{z}}\mu_0/2\pi d$. The force $\boldsymbol{F}_1$ on wire 1 is therefore $\boldsymbol{F}_1 = I\boldsymbol{l} \times \boldsymbol{B}_2$, where ℓ points along $\hat{\boldsymbol{y}}$. The force per length is therefore

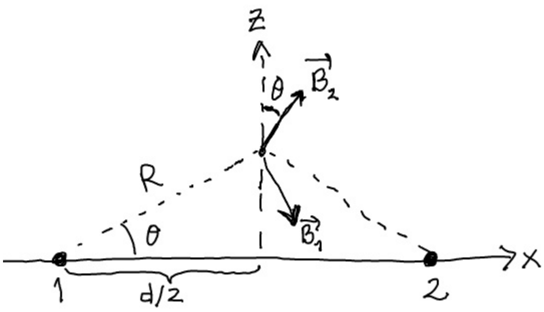
$$\frac{\boldsymbol{F}_1}{\ell} = IB_2(\hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}}) = \frac{\mu_0 I^2}{2\pi d} \hat{\boldsymbol{x}}. \tag{15}$$

The force is attractive since it points towards wire 2.

(b) The equation implicitly refers to some volume, let's call it Ω. The first term is the total electromagnetic force on the charges inside Ω. The second term is an integral over the surface of Ω, where $\overleftrightarrow{\boldsymbol{T}}$ is the Maxwell stress tensor of the electromagnetic fields, which has an interpretation as force per unit area (with diagonal elements representing pressures and off-diagonal elements representing shears).

(Also, $-\overleftrightarrow{\boldsymbol{T}}$ can be interpreted as a momentum current density.) The third term is the time derivative of the total momentum stored in the electromagnetic fields inside Ω (the integrand $\boldsymbol{S}/c^2$ is the momentum density of the fields).

(c) We pick the volume Ω to be the "half-space" $x < 0$, as it contains all of wire 1 and none of wire 2, and since the boundary plane $x = 0$ lies symmetrically between the wires. Since the problem is static, the term with the time derivative in the force equation vanishes, giving $\boldsymbol{F} = \oint \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a}$. As hinted in the text, let us consider the contribution of the plane $x = 0$ to the surface integral. The Maxwell stress tensor only has magnetic contributions:[1]



$$T_{ij} = \frac{1}{\mu_0}(B_i B_j - \frac{1}{2}\delta_{ij} B^2). \tag{16}$$

It is important to note that in this expression, $\boldsymbol{B}$ is the *total* magnetic field, i.e. $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$. Of course, $B_y = 0$ everywhere. Furthermore, in the plane $x = 0$ we also have (see the figure) $B_z = 0$ (the z components from wires 1 and 2 cancel each other) and $B_x = 2B_1 \sin\theta$ where $B_1 = B_2$ is the magnitude of the individual fields and θ is the angle defined in the figure. Here (see the figure)

$$B_1 = \frac{\mu_0 I}{2\pi R} \quad \text{where} \quad \cos\theta = \frac{d/2}{R} \quad \Rightarrow \quad B_x = \frac{2\mu_0 I \cos\theta \sin\theta}{\pi d}. \tag{17}$$

In the surface integral we have

$$\overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} = T_{ij}\hat{\boldsymbol{x}}_i\hat{\boldsymbol{x}}_j \cdot da_k\hat{\boldsymbol{x}}_k = T_{ij}da_k\hat{\boldsymbol{x}}_i \underbrace{(\hat{\boldsymbol{x}}_j \cdot \hat{\boldsymbol{x}}_k)}_{\delta_{jk}} = T_{ij}da_j\hat{\boldsymbol{x}}_i. \tag{18}$$

For the plane $x = 0$ (the yz plane), $da_j = da\,\delta_{jx}$, giving

$$\overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} = T_{ix}da\,\hat{\boldsymbol{x}}_i = da(T_{xx}\hat{\boldsymbol{x}} + T_{yx}\hat{\boldsymbol{y}} + T_{zx}\hat{\boldsymbol{z}}) \tag{19}$$

where da is an infinitesimal surface element in the yz plane (i.e. $da = dy\,dz$ in Cartesian coordinates). Since $B_y = B_z = 0$ in this plane, it follows that

$$T_{yx} = T_{zx} = 0 \quad \text{and} \quad T_{xx} = \frac{1}{\mu_0}(B_x^2 - \frac{1}{2}B_x^2) = \frac{B_x^2}{2\mu_0}. \tag{20}$$

This gives,

$$\int_{x=0} \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} \;\; = \;\; \hat{\boldsymbol{x}}\int_{x=0} da\; T_{xx} = \hat{\boldsymbol{x}}\frac{2\mu_0 I^2}{(\pi d)^2}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\;\cos^2\theta\sin^2\theta. \tag{21}$$

Since we want the force per unit length we divide by the y-integral. Let us also change integration variable from z to θ, using (see figure) $z = (d/2)\tan\theta$, so $dz = (d/2)(\cos^2\theta)^{-1}d\theta$. This gives

$$\frac{\int_{x=0} \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a}}{\ell} = \hat{\boldsymbol{x}}\frac{2\mu_0 I^2}{(\pi d)^2} \cdot \frac{d}{2}\int_{-\pi/2}^{\pi/2} d\theta\;\sin^2\theta. \tag{22}$$

The integral is just $\langle \sin^2 \rangle = 1/2$ multiplied by the integration length π. Thus

$$\frac{\int_{x=0} \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a}}{\ell} = \frac{\mu_0 I^2}{2\pi d}\hat{\boldsymbol{x}}, \tag{23}$$

[1]The electric field is zero everywhere outside the wires. (Note however that the electric field is nonzero inside the wires if the wires are not perfect conductors, since $\boldsymbol{E} = \boldsymbol{J}/\sigma$ by Ohm's law.)

which is exactly the result we got for the force per unit length in (a). Indeed, it can be argued that the contributions to the surface integral from the other boundaries "at infinity" vanish (we don't go into more details here).

Problem 3

(a) The wave equations follow from the two Maxwell equations containing sources ρ and $\boldsymbol{J}$ (the other two Maxwell equations are automatically satisfied by the potentials). The Ampere-Maxwell law $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0\epsilon_0 \partial \boldsymbol{E}/\partial t$ becomes, when using $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, $\boldsymbol{E} = -\nabla V - \partial \boldsymbol{A}/\partial t$, and $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$ (all these formulas are given in the formula set), and rearranging a little,

$$\nabla \left(\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} \right) - \nabla^2 \boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \mu_0 \boldsymbol{J}, \tag{24}$$

(we also used that $\mu_0\epsilon_0 = 1/c^2$). We see that the desired wave equation for $\boldsymbol{A}$ follows if we choose

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0, \tag{25}$$

which therefore is the Lorenz gauge condition. Similarly, Gauss's law $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$ becomes

$$-\nabla^2 V - \frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = \frac{\rho}{\epsilon_0}. \tag{26}$$

Eq. (25) gives $\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = -\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$, which when inserted into (26) gives the desired wave equation for V.

(b) While the original equation contains partial derivatives wrt z and t, the rewritten equation instead contains partial derivatives wrt $\xi = z - vt$. Thus we need to rewrite the former in terms of the latter. Using the chain rule we have

$$\frac{\partial}{\partial z} = \frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi} \quad \Rightarrow \quad \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial \xi^2}, \tag{27}$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} = -v\frac{\partial}{\partial \xi} \quad \Rightarrow \quad \frac{\partial^2}{\partial t^2} = -v\frac{\partial}{\partial \xi}\left(-v\frac{\partial}{\partial \xi}\right) = v^2\frac{\partial^2}{\partial \xi^2}. \tag{28}$$

This gives

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2 V}{\partial \xi^2} = (1-\beta^2)\frac{\partial^2 V}{\partial \xi^2}. \tag{29}$$

Also using the expression for $\rho(\boldsymbol{r}, t)$, in which $\delta(z - vt) = \delta(\xi)$, gives the desired differential equation.

(c) We have

$$\frac{\partial}{\partial \xi} = \frac{dz'}{d\xi}\frac{\partial}{\partial z'} = \gamma\frac{\partial}{\partial z'} \quad \Rightarrow \quad \frac{\partial^2}{\partial \xi^2} = \gamma\frac{\partial}{\partial z'}\gamma\frac{\partial}{\partial z'} = \gamma^2\frac{\partial^2}{\partial z'^2}. \tag{30}$$

Using also that $(1-\beta^2) = \gamma^{-2}$ gives $(1-\beta^2)\partial^2 V/\partial \xi^2 = \gamma^{-2}\gamma^2\partial^2 V/\partial z'^2 = \partial^2 V/\partial z'^2$. Furthermore, using the identity $\delta(ax) = |a|^{-1}\delta(x)$ in the formula set gives $\delta(\xi) = \delta(z'/\gamma) = |\gamma|\delta(z') = \gamma\delta(z')$. Putting these things together, we arrive at the differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z'^2} = -\frac{\gamma q}{4\pi\epsilon_0}\delta(x)\delta(y)\delta(z'). \tag{31}$$

We recognize this as the Poisson equation for a point charge $\gamma q \equiv Q$ at the origin $(0,0,0)$ in a space with Cartesian coordinates (x, y, z'). We know that the potential for this problem is $V = Q/4\pi\epsilon_0 r'$ where $r' \equiv \sqrt{x^2 + y^2 + z'^2}$. Inserting $Q = \gamma q$ and $z' = \gamma\xi = \gamma(z - vt)$ then gives the desired expression for V.

(d) The wave equations for V and A_z only differ on the rhs, containing ρ/ϵ_0 and $\mu_0 J_z$, respectively. Using $J_z = \rho v$, we see that ρ is a common factor, so the wave equation for A_z is obtained by making the replacement $1/\epsilon_0 \to \mu_0 v$ in the wave equation for V. As this replacement just involves constants, A_z is found by making the same replacement in the solution for V, giving

$$A_z(x, y, z, t) = \frac{\mu_0 \gamma q v}{4\pi\sqrt{x^2 + y^2 + \gamma^2(z - vt)^2}}. \tag{32}$$

(e) The electric field is $\boldsymbol{E} = -\nabla V - \partial \boldsymbol{A}/\partial t$ (cf. formula set). Using the solutions for V and $\boldsymbol{A}$ gives

$$\nabla V = \hat{\boldsymbol{x}}\frac{\partial V}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial V}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial V}{\partial z} = -\frac{\gamma q}{4\pi\epsilon_0}\frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + \gamma^2(z - vt)\hat{\boldsymbol{z}}}{[x^2 + y^2 + \gamma^2(z - vt)^2]^{3/2}}, \tag{33}$$

$$\frac{\partial \boldsymbol{A}}{\partial t} = \hat{\boldsymbol{z}}\frac{\partial A_z}{\partial t} = -\frac{\mu_0 \gamma q v}{4\pi}\frac{\gamma^2(z - vt)(-v)\hat{\boldsymbol{z}}}{[x^2 + y^2 + \gamma^2(z - vt)^2]^{3/2}}. \tag{34}$$

Combining these results to find $\boldsymbol{E}$, one sees that the x and y components take the form stated in the text. For the z component we need to explicitly sum the contributions from (33) and (34), giving

$$\frac{\gamma q}{4\pi}\frac{z - vt}{[x^2 + y^2 + \gamma^2(z - vt)^2]^{3/2}}\left\{\frac{\gamma^2}{\epsilon_0} - \mu_0\gamma^2 v^2\right\}. \tag{35}$$

Using now that $\mu_0 = 1/\epsilon_0 c^2$, the quantity in curly brackets becomes $(1/\epsilon_0)\gamma^2(1 - v^2/c^2) = 1/\epsilon_0$, thus also giving the z component stated in the text.

(f) We note that $\boldsymbol{R} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + (z - vt)\hat{\boldsymbol{z}} = R\hat{\boldsymbol{R}}$ and that from Fig. 3 it follows that $z - vt = R\cos\theta$.[2] Thus $R^2 = x^2 + y^2 + (z - vt)^2 = x^2 + y^2 + R^2\cos^2\theta$, i.e. $x^2 + y^2 = R^2(1 - \cos^2\theta) = R^2\sin^2\theta$. Thus

$$\boldsymbol{E} = \frac{\gamma q}{4\pi\epsilon_0}\frac{R\hat{\boldsymbol{R}}}{[R^2\sin^2\theta + \gamma^2R^2(1 - \sin^2\theta)]^{3/2}} = \frac{\gamma q}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{R}}}{R^2}\frac{1}{[\gamma^2 + (1 - \gamma^2)\sin^2\theta]^{3/2}}. \tag{36}$$

Next, pulling out $\gamma^3 = (\gamma^2)^{3/2}$ in the denominator will give us the desired $\gamma^{-2} = 1 - \beta^2$ in the numerator. Left inside the square brackets is then $1 + [(1 - \gamma^2)/\gamma^2]\sin^2\theta$, where $(1 - \gamma^2)/\gamma^2 = \gamma^{-2} - 1 = 1 - \beta^2 - 1 = -\beta^2$. This gives the expression stated in the text.

(g) For arbitrary β one can see that

$$\text{for } \theta = 0, \pi \text{ (forward/backward directions)}, \quad \boldsymbol{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{R}}}{R^2}(1 - \beta^2), \tag{37}$$

$$\text{for } \theta = \pi/2 \text{ (transverse directions)}, \quad \boldsymbol{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{R}}}{R^2}\frac{1}{\sqrt{1 - \beta^2}}. \tag{38}$$

Ultrarelativistic case ($\beta \approx 1$): $|\boldsymbol{E}|$ is very small in the forward/backward directions ($\to 0$ as $\beta \to 1$) and very large in the transverse directions ($\to \infty$ as $\beta \to 1$). The variation of $|\boldsymbol{E}|$ with the angle θ is therefore very strong in this case.
Nonrelativistic case ($\beta \approx 0$): For both the forward/backward and transverse directions, $\boldsymbol{E}$ is approximately given by the Coulomb form $q/4\pi\epsilon_0 R^2$. The variation of $|\boldsymbol{E}|$ with the angle θ is therefore very weak in this case.

(h) The energy per unit time is $P = \oint \boldsymbol{S} \cdot d\boldsymbol{a}$, where $\boldsymbol{S} = \mu_0^{-1}(\boldsymbol{E} \times \boldsymbol{B})$ is the Poynting vector and the integral is over the surface of the sphere. Because $\boldsymbol{E} \parallel \hat{\boldsymbol{R}}$, it follows from the properties of the cross product that $\boldsymbol{S}$ has no component along $\hat{\boldsymbol{R}}$. Since $d\boldsymbol{a} \parallel \hat{\boldsymbol{R}}$, it follows that $\boldsymbol{S} \cdot d\boldsymbol{a} = 0$ everywhere on the spherical surface, and therefore $P = 0$.

(i) Radiation is electromagnetic energy that escapes from a source and propagates "to infinity". It can be more precisely defined by considering the energy per unit time $P(r) = \oint \boldsymbol{S} \cdot d\boldsymbol{a}$ passing through a sphere of radius r centered at the source. If $\lim_{r\to\infty} P(r) \neq 0$ the source radiates. Since the area of the sphere increases as r^2, radiation requires S to fall off no faster than $1/r^2$, which in turn requires E and B to have contributions decaying like $1/r$. Radiation requires accelerating charges. Thus the particle with constant velocity does not radiate. (This is also consistent with the results found in (f) and (h)).

[2] Alternatively, this can be seen by calculating $\boldsymbol{R} \cdot \hat{\boldsymbol{z}}$, which on the one hand equals $R \cdot 1 \cdot \cos\theta$, and on the other hand equals $(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + (z - vt)\hat{\boldsymbol{z}}) \cdot \hat{\boldsymbol{z}} = z - vt$.