# Solution for exam TFY4240, June 2017

## Problem 1

(a) $\nabla \cdot \vec{D} = \rho_f$ and $\vec{D} = \epsilon \vec{E}$

$\Rightarrow \quad \nabla \cdot (\epsilon \vec{E}) = \rho_f$

Inside a medium $m$, $\epsilon = \epsilon_m$ (a constant)

$\Rightarrow \quad \epsilon_m \nabla \cdot \vec{E} = \rho_f$

In electrostatics, $\vec{E} = -\nabla V$

$$\Rightarrow \quad \underline{\underline{\nabla^2 V = -\rho_f / \epsilon_m}}$$

(b) $\vec{E} = -\nabla V \quad \Rightarrow \quad dV = -\vec{E} \cdot d\vec{r}$

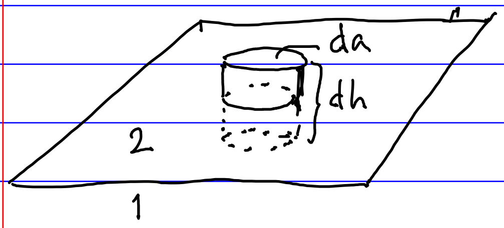
Integrating this from a point $\vec{r}_1$ in medium 1 to a point $\vec{r}_2$ in medium 2 gives

$$V_2 - V_1 = \int_1^2 dV = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r} \qquad \text{(path-indep.)}$$

Letting $\vec{r}_1$ and $\vec{r}_2$ approach the same point on the boundary (i.e. $\vec{r}_2 - \vec{r}_1 \to 0$), the integral $\to 0$ since $\vec{E}$ is finite

$$\Rightarrow \quad \underline{\underline{V_2 - V_1 = 0}}$$

Next, we introduce a Gaussian "pillbox"-shaped surface which straddles the boundary, as shown in the figure below:




Gauss's law on integral form:

$$\oint_A \vec{D} \cdot d\vec{a} = Q_{f,\text{inside}}$$

Considering the limit of a "wafer-thin" box (height $dh \to 0$)

(1) The only contribution to the surface integral comes from the top & bottom parts (parallel to boundary), each of area $da$.

$$\Rightarrow \oint \vec{D} \cdot d\vec{a} = (\vec{D}_2 - \vec{D}_1) \cdot \hat{n}\, da = (\epsilon_2 \vec{E}_2 - \epsilon_1 \vec{E}_1) \cdot \hat{n}\, da$$

(2) The only contribution to $Q_{f,\text{inside}}$ can come from free charge at the boundary, with surface charge density $\sigma_f$:

$$\Rightarrow Q_{f,\text{inside}} = \sigma_f\, da$$

Cancelling $da$ gives $(\epsilon_2 \vec{E}_2 - \epsilon_1 \vec{E}_1) \cdot \hat{n} = \sigma_f$

Using $\vec{E} = -\nabla V$ and $\partial_n V = \hat{n} \cdot \nabla V$ then gives

$$\underline{\underline{\epsilon_2 \partial_n V_2 - \epsilon_1 \partial_n V_1 = -\sigma_f}}$$

(c) As $r \to \infty$, $\vec{E} = -\nabla V \to \vec{E}_0 = E_0 \hat{z}$

$\Rightarrow V \to -E_0 z + C$ as $r \to \infty$

We take $C = 0$ so that in the absence of $\vec{E}_0$, this boundary condition for $V$ reduces to $V \to 0$. As $z = r\cos\theta$ we thus get

$$V(\vec{r}) \to -E_0 r \cos\theta = -E_0 r P_1(\cos\theta) \quad \text{as } r \to \infty$$

Comparing this with the expansion gives (o = outside)

$$A_1^{(o)} = -E_0 \quad \text{and} \quad A_\ell^{(o)} = 0 \quad \text{for } \ell \neq 1$$

$$\Rightarrow V_{(o)}(\vec{r}) = -E_0 r\cos\theta + \sum_\ell \frac{B_\ell^{(o)}}{r^{\ell+1}} P_\ell(\cos\theta)$$

In the expansion of $V_{(i)}(\vec{r})$ (i = inside), $B_\ell^{(i)}$ must be 0 for all $\ell$ to prevent $V_i(\vec{r})$ from diverging as $r \to 0$. Thus

$$V_{(i)}(\vec{r}) = \sum_{\ell=0}^{\infty} A_\ell^{(i)} r^\ell P_\ell(\cos\theta)$$

In the following, rename $A_\ell^{(i)} \equiv A_\ell$ and $B_\ell^{(o)} \equiv B_\ell$. To find these we must use Eqs. (2) and (3) in the text (with $m = 1$ inside, $m = 2$ outside)

First Eq. (2): $V_{(o)} = V_{(i)}$ for $r = R$

$$\Rightarrow -E_0 R\cos\theta + \sum_\ell \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) = \sum_\ell A_\ell R^\ell P_\ell(\cos\theta)$$

Equating coefficients of $P_\ell$ for each $\ell$ gives

$$\ell = 1: \quad -E_0 R + \frac{B_1}{R^2} = A_1 R \Rightarrow \underline{B_1 = R^3 (A_1 + E_0)} \quad (*)$$

$$\ell \neq 1: \quad \frac{B_\ell}{R^{\ell+1}} = A_\ell R^\ell \Rightarrow \underline{B_\ell = R^{2\ell+1} A_\ell} \quad (**)$$

Next, for Eq. (3), note that $\partial_n = \partial_r$. We need

$$\partial_r V_{(o)} = -E_0 \cos\theta + \sum_\ell (-1)(\ell+1) \frac{B_\ell}{r^{\ell+2}} P_\ell(\cos\theta)$$

$$\partial_r V_{(i)} = \sum_\ell A_\ell \, \ell \, r^{\ell-1} P_\ell(\cos\theta)$$

Thus Eq. (3) gives (as $\sigma_f = 0$) $\epsilon \partial_r V_{(o)}\big|_{r=R} = \epsilon_0 \partial_r V_{(i)}\big|_{r=R}$

$$\Rightarrow \epsilon\left[-E_0 P_1 - \sum_\ell \frac{B_\ell(\ell+1)}{R^{\ell+2}} P_\ell\right] = \epsilon_0 \sum_\ell A_\ell \, \ell \, R^{\ell-1} P_\ell$$

Equating coefficients of $P_\ell$ for each $\ell$ gives

$$\ell = 1: \quad \epsilon\left(-E_0 - \frac{2B_1}{R^3}\right) = \epsilon_0 A_1$$

$$\ell \neq 1: \quad -\epsilon \frac{B_\ell(\ell+1)}{R^{\ell+2}} = \epsilon_0 A_\ell \, \ell \, R^{\ell-1}$$

Using $(*)$ the $\ell = 1$ equation gives

$$-\epsilon\left[E_0 + \frac{2}{\cancel{R^3}} \cancel{R^3}(A_1 + E_0)\right] = \epsilon_0 A_1$$

$$\Rightarrow \quad \underline{\underline{A_1 = -\frac{3\epsilon}{\epsilon_0 + 2\epsilon} E_0}} \quad \Rightarrow \quad \underline{\underline{B_1 = \frac{\epsilon_0 - \epsilon}{\epsilon_0 + 2\epsilon} E_0 R^3}}$$

Using $(**)$ the $\ell \neq 1$ equation gives

$$-\epsilon \frac{\cancel{R^{2\ell+1}} A_\ell(\ell+1)}{\cancel{R^{\ell+2}}} = \epsilon_0 A_\ell \, \ell \, \cancel{R^{\ell-1}} \quad \Rightarrow \underline{A_\ell = 0} \Rightarrow \underline{B_\ell = 0}$$

$$\Rightarrow \quad \underline{\underline{V_{\text{outside}}(\vec{r}) = -E_0 r\cos\theta - \frac{\epsilon - \epsilon_0}{2\epsilon + \epsilon_0} \frac{E_0 R^3}{r^2} \cos\theta}}$$

(The first term is due to the external field, the second term is a dipole potential)

$$V_{inside}(\vec{r}) = -\frac{3\epsilon}{\epsilon_0 + 2\epsilon} E_0 \, r\cos\theta$$

As $r\cos\theta = z$, we evaluate $\vec{E}_{inside} = -\nabla V_{inside}$ using cartesian coordinates

$$\Rightarrow \vec{E}_{inside}(\vec{r}) = \frac{3\epsilon}{\epsilon_0 + 2\epsilon} E_0 \, \hat{z} \qquad \text{(constant)}$$

(Introducing the dielectric constant $\kappa \equiv \epsilon/\epsilon_0$ $(>1)$, we see that

$$E_{inside} = \frac{3\kappa}{1+2\kappa} E_0 = \frac{3}{1/\kappa + 2} E_0 > E_0 \;)$$

## Problem 2

(a) Eq. (5) is a statement of conservation of energy for a volume $\Omega$ bounded by a surface $a$, inside of which there can be electric charges (also known as Poynting's theorem).

$U_{em}$ is the energy stored in the electromagnetic fields inside $\Omega$. Thus $dU_{em}/dt$ is the rate of change of this energy.

$\oint \vec{S} \cdot d\vec{a}$ is the rate at which energy is carried out of $\Omega$ (by passing out through the surface $a$) by the electromagnetic fields (here $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$ is the Poynting vector)

$dW/dt$ is the rate at which work is done on the charges inside $\Omega$ by the fields (really just by $\vec{E}$, since $\vec{B}$ does no work)

In words, the electromagnetic energy inside $\Omega$ can decrease due to energy flowing out and due to work being done on the charges (which increases their mechanical (kinetic) energy).

(b) $\vec{j} = \sigma \vec{E} \Rightarrow \underline{\underline{\vec{E}}} = \frac{\vec{j}}{\sigma} = \underline{\underline{\frac{I}{\sigma \pi b^2} \hat{z}}}$

To find $\vec{B}$ we use Ampere's law on integral form on a circular loop $s$ = constant. $\vec{B}$ will be tangential to this loop, with a constant magnitude $B$ and a direction given by a right-hand rule:

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{inside} \Rightarrow B \cdot 2\cancel{\pi} \cancel{s} = \mu_0 j \cancel{\pi} s^{\cancel{2}}$$

$$\Rightarrow B = \frac{\mu_0 j s}{2} = \frac{\mu_0 I}{2\pi b} \frac{s}{b} \Rightarrow \underline{\underline{\vec{B} = \frac{\mu_0 I}{2\pi b}\left(\frac{s}{b}\right)\hat{\phi}}}$$

(c) $$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{1}{\cancel{\mu_0}} \frac{I}{\sigma \pi b^2} \frac{\cancel{\mu_0} I s}{2\pi b^2} \underbrace{(\hat{z} \times \hat{\phi})}_{-\hat{s}} = -\frac{I^2 s}{2\pi^2 \sigma b^4} \hat{s}$$

$\Rightarrow \vec{S}$ points radially, towards the wire axis

Thus $\oint_a \vec{S} \cdot d\vec{a}$ gets no contribution from the flat (top & bottom) surfaces of $a$, as $\vec{S} \cdot \hat{n} = 0$ there.

Thus the only contribution is from the curved surface where $s = b$ and $\hat{n} = \hat{s}$

$$\Rightarrow \oint_a \vec{S} \cdot d\vec{a} = -L \cdot \cancel{2\pi b} \cdot \frac{I^2}{\cancel{2}\pi^{\cancel{2}} \sigma b^{\cancel{3}2}} = -\frac{L}{\sigma \pi b^2} \cdot I^2$$

As the fields are static, $dU_{em}/dt = 0 \Rightarrow \underline{\underline{\frac{dW}{dt} = \frac{L}{\sigma \pi b^2} I^2}}$

This equals $RI^2$ (power loss due to ohmic heating), as $V = RI = EL \Rightarrow R = EL/I = L/(\sigma \pi b^2)$

## Problem 3.

(a) Due to the spherical symmetry of the charge distribution, it is natural to introduce spherical coordinates $(r, \theta, \phi)$, with $\rho$ independent of $\theta$ and $\phi$. Also, $\rho$ is clearly only nonzero on the shell, i.e. for $r = R(t)$. This suggests that $\rho$ can be written as $\rho = f(r,t)\,\delta(r - R(t))$ where the function $f(r,t)$ is to be determined. Actually, due to the delta function which requires $r = R(t)$ to be nonzero, we can write $f(r,t)\,\delta(r-R(t)) = f(R(t),t)\,\delta(r-R(t))$ $\equiv g(t)\,\delta(r-R(t))$ where $g(t) = f(R(t),t)$. To determine $g(t)$ we use that the total charge $Q$ is conserved:

$$Q = \int d^3r\, \rho(\vec{r},t) = \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta) \int_0^{\infty} dr\, r^2\, g(t)\,\delta(r-R(t))$$

$$= 2\pi \cdot 2 \cdot g(t) \cdot R^2(t) \Rightarrow g(t) = \frac{Q}{4\pi R^2(t)}$$

$$\Rightarrow \rho(\vec{r},t) = \frac{Q}{4\pi R^2(t)}\,\delta(r - R(t))$$

$$\Rightarrow \underline{\underline{\rho(\vec{r},t) = \frac{Q}{4\pi r^2}\,\delta(r-R(t))}} \qquad (\text{as } \delta(\cdot\cdot) = 0 \text{ for } r \neq R(t))$$

(b) We can write $\vec{E} = E_r\,\hat{r} + E_\theta\,\hat{\theta} + E_\phi\,\hat{\phi}$. Due to the spherical symmetry of the problem, $E_\theta$ and $E_\phi$ must be $0$, and the only remaining nonzero component $E_r$ must be independent of $\theta$ and $\phi$. Therefore $\vec{E}(\vec{r},t) = E_r(r,t)\,\hat{r}$. It then follows from the formula for the curl in spherical coordinates that $\underline{\nabla \times \vec{E} = 0}$

Thus we have $\nabla \cdot \vec{E} = \rho/\epsilon_0$ (Gauss's law)

$\nabla \times \vec{E} = 0$

The 2nd equation implies that this set of equations for the spatial structure of $\vec{E}$ takes the same form as for an electrostatics problem (the time $t$ is mathematically just an "innocuous" parameter in $\rho$ that will make $\vec{E}$ time-dependent). Given the spherical symmetry it is then natural to solve the problem using Gauss's law on integral form obtained from integrating $\nabla \cdot \vec{E} = \rho/\epsilon_0$ over a volume $\Omega$ enclosed by a "Gaussian" surface $a$ and then using the divergence theorem (D.T.):

$$\int_\Omega d^3r\, \nabla \cdot \vec{E} \underset{\text{D.T.}}{=} \oint_a d\vec{a} \cdot \vec{E} = \frac{1}{\epsilon_0} \int_\Omega d^3r\, \rho = \frac{Q_{\text{inside}}}{\epsilon_0}$$

At this stage we exploit the symmetry by taking the Gaussian surface to be spherical (with radius $r$) and centered at the origin, as $E_r$ is constant on this surface. Thus

$$\epsilon_0 \oint_a \vec{E} \cdot d\vec{a} = \epsilon_0 E_r 4\pi r^2 = Q_{\text{inside}} = \begin{cases} Q & \text{if } r > R(t) \\ 0 & \text{if } r < R(t) \end{cases}$$

$$= Q\,\Theta(r - R(t)) \Rightarrow \underline{\underline{\vec{E}(\vec{r},t) = \frac{Q}{4\pi\epsilon_0 r^2}\,\Theta(r - R(t))\,\hat{r}}}$$

Here $\Theta(u)$ is the (Heaviside) step function

$$\Theta(u) \equiv \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{if } u < 0 \end{cases}$$

(not to be confused with the angle $\theta$!)

(c) $$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t} \frac{Q}{4\pi r^2} \delta(r - R(t)) = \frac{Q}{4\pi r^2} \frac{\partial}{\partial t} \delta(r - R(t))$$

Define $u = r - R(t)$ and use the chain rule:

$$\frac{\partial}{\partial t} \delta(u) = \frac{\partial u}{\partial t} \frac{d}{du} \delta(u) = -\dot{R}(t) \frac{d}{du} \delta(u)$$

$$\Rightarrow \underline{\frac{\partial \rho}{\partial t} = -\frac{Q \dot{R}(t)}{4\pi r^2} \frac{d}{du} \delta(u)}$$

As the spatial dependence of $\vec{j}(\vec{r}, t)$ takes the form $j_r \hat{r}$, we see from the formula set that

$$\nabla \cdot \vec{j} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 j_r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{Q \dot{R}(t)}{4\pi} \delta(r - R(t)) \right)$$

$$= \frac{Q \dot{R}(t)}{4\pi r^2} \frac{\partial}{\partial r} \delta(r - R(t))$$

$$\frac{\partial}{\partial r} \delta(u) = \frac{\partial u}{\partial r} \frac{d}{du} \delta(u) = 1 \cdot \frac{d}{du} \delta(u) = \frac{d}{du} \delta(u)$$

$$\Rightarrow \underline{\nabla \cdot \vec{j} = \frac{Q \dot{R}(t)}{4\pi r^2} \frac{d}{du} \delta(u)}$$

Comparing the underlined expressions for $\frac{\partial \rho}{\partial t}$ and $\nabla \cdot \vec{j}$ shows that the continuity equation is satisfied

(d) Because of the spherical symmetry (the current density $\vec{j}(\vec{r}, t) = j_r(r, t)\,\hat{r}$) it follows that $\vec{B}$ must take the form $\vec{B}(\vec{r}, t) = B_r(r, t)\,\hat{r}$. Thus the formula for the curl in spherical coordinates gives $\underline{\nabla \times \vec{B} = 0}$. Alternatively, this result can also be shown from the Ampere-Maxwell law

$$\nabla \times \vec{B} = \mu_0 \left( \vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

We know $\vec{j}$, so it remains to find $\epsilon_0 \frac{\partial \vec{E}}{\partial t}$ :

$$\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \epsilon_0 \frac{\partial}{\partial t} \frac{Q}{4\pi\epsilon_0 r^2} \theta(r - R(t))\, \hat{r}$$

$$= \frac{Q\hat{r}}{4\pi r^2} \frac{\partial}{\partial t} \theta(r - R(t)), \quad \text{where}$$

$$\frac{\partial}{\partial t} \theta(r - R(t)) = \frac{\partial u}{\partial t} \frac{d}{du} \theta(u) = -\dot{R}(t)\,\delta(u)$$

$$\Rightarrow \epsilon_0 \frac{\partial \vec{E}}{\partial t} = -\frac{Q\dot{R}(t)}{4\pi r^2} \delta(r - R(t))\, \hat{r} = -\vec{j} \Rightarrow \nabla \times \vec{B} = 0$$

Thus $\vec{B}$ obeys $\nabla \cdot \vec{B} = 0$ (Gauss's law for $\vec{B}$)

$$\nabla \times \vec{B} = 0$$

These equations take the same form as those for $\vec{E}$, with $\rho$ replaced by $0$ ($=$ "$\rho_{magnetic}$"). Using the same method to find $\vec{B}$ thus gives

$$\underline{\underline{\vec{B}(\vec{r}, t) = 0}}$$

(e) To consider radiation we consider a spherical surface $a$ of radius $r$ centered at the origin. The time-average of the energy passing through the surface per unit time is $P(r) = \oint_a d\vec{a} \cdot \langle \vec{S} \rangle$

where $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$ and $\langle \vec{S} \rangle$ its time average. The shell radiates if $\lim_{r \to \infty} P(r) \neq 0$.

Since $\vec{B} = 0$, $\vec{S} = \vec{0}$, so there is <u>no radiation</u>. (Alternatively, the conclusion of no radiation can also be argued from $\vec{E} \parallel \hat{r} \Rightarrow \vec{S} \perp \hat{r} \Rightarrow \vec{S} \cdot d\vec{a} = 0$. The fact that $|\vec{E}| \propto 1/r^2$ is also inconsistent with radiation.)