NTNU **Institutt for fysikk**

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Exam in TFY4240 Electromagnetic Theory

May 22, 2024
09:00–13:00

Allowed help: Alternativ **C**

A permitted basic calculator and a mathematical formula book (Rottmann or equivalent).

This problem set consists of 9 pages.

This exam consists of 3 problems, each containing several subproblems. There are, in total, 10 subproblems. Each subproblem (1a-1b-...) will be given equal weight in the grading.

I will be available for questions related to the problems themselves (though not the answers!). I plan to do one round around 10.30.

The problems are given in English only. Do not hesitate to ask if you have any language problems related to the exam set. You are free to use either English or Norwegian for your answers.

Some formulas are given in the appendix on the pages following the last problem.

Good luck!

Problem 1.

a) Demonstrate that charge conservation follows from Maxwell‘s equations.

Solution

We compute the rate of change of Gauss‘s law (70) and make use of Ampere‘s law (73):

$$\partial_t \boldsymbol{\nabla} \cdot \boldsymbol{D} = \partial_t \rho_f \,, \tag{1}$$

$$\boldsymbol{\nabla} \cdot \partial_t \boldsymbol{D} = \partial_t \rho_f \,, \tag{2}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{H} - \boldsymbol{J}_f) = \partial_t \rho_f \,. \tag{3}$$

Using $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}) = 0$, we find

$$\partial_t \rho_f + \boldsymbol{\nabla} \cdot \boldsymbol{J}_f = 0 \,, \tag{4}$$

which is the continuity equation that reflects charge conservation. The charge current out of any small volume element compensates for the rate of change of the charge within the same volume.

b) Demonstrate that when we express the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ as

$$\boldsymbol{E} = -\boldsymbol{\nabla} V - \partial_t \boldsymbol{A} \,, \tag{5}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \,, \tag{6}$$

in terms of the scalar potential V and the vector potential $\boldsymbol{A}$, then, two of Maxwell‘s equations are automatically satisfied.

Solution

We consider the Gauss‘s law for the magnetic induction (71) and insert the expression (6)

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \,, \tag{7}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0 \,, \tag{8}$$

to see that this expression is automatically satisfied because $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$ for any function $\boldsymbol{A}$.

Next, we consider Faraday‘s law and insert the expressions (5) and (6):

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \,, \tag{9}$$

$$\boldsymbol{\nabla} \times (-\boldsymbol{\nabla} V - \partial_t \boldsymbol{A}) = -\partial_t \boldsymbol{\nabla} \times \boldsymbol{A} \,, \tag{10}$$

$$-\partial_t \boldsymbol{\nabla} \times \boldsymbol{A} = -\partial_t \boldsymbol{\nabla} \times \boldsymbol{A} \,, \tag{11}$$

which is always satisifed, where we used that $\boldsymbol{\nabla} \times \boldsymbol{\nabla} V = 0$.

c) Find the equations the scalar potential V and the vector potential $\boldsymbol{A}$ introduced in (5) and (6) must satisfy in linear and isotropic materials.

Solution

We use the constitute relation $\boldsymbol{D} = \epsilon \boldsymbol{E}$ and Gauss's law (70) to find

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho_f/\epsilon\,, \tag{12}$$

$$\boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla} V - \partial_t \boldsymbol{A}) = \rho_f/\epsilon\,, \tag{13}$$

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{v^2}\partial_t^2\right) V = -\rho_f/\epsilon - \partial_t \left(\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{v^2}\partial_t V\right)\,, \tag{14}$$

where the velocity of light in the material is $v = 1/\sqrt{\epsilon\mu}$.

Next, we use the constitute relations $\boldsymbol{B} = \mu \boldsymbol{H}$ and $\boldsymbol{D} = \epsilon \boldsymbol{E}$ with Ampere's law (73) to find

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial D}{\partial t}\,, \tag{15}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \mu \boldsymbol{J}_f + \epsilon\mu\partial_t\left(-\boldsymbol{\nabla} V - \partial_t \boldsymbol{A}\right)\,, \tag{16}$$

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{v^2}\partial_t^2\right) \boldsymbol{A} = -\mu \boldsymbol{J}_f + \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \boldsymbol{A} + \frac{1}{v^2}\partial_t V\right)\,. \tag{17}$$

Problem 2.

We consider a vacuum-metal structure. In a Cartesian coordinate system with unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, the vacuum exists for all x and y when $z > 0$. The metal is for all x and y when $z < 0$. The vacuum-metal interface is at $z = 0$.

Inside the metal, Ohm's law applies

$$\boldsymbol{J}_f = \sigma \boldsymbol{E}\,, \tag{18}$$

where σ is the conductivity, $\boldsymbol{E}$ is the electric field, and $\boldsymbol{J}_f$ is the free charge current density. There are no free charges, $\rho_f = 0$. There are also no free surface charges and no free surface charge currents.

We consider an incident TE (transverse electric) wave from the vacuum side that can be reflected off the metal. We assume the wave propagates straight towards the interface. With these assumptions, the wave depends on the spatial coordinate z and the time t. We use complex notation. On the vacuum side, the wave is then

$$\boldsymbol{E}_v(z,t) = E_0 \hat{\mathbf{x}} \left[e^{i(kz-\omega t)} + r e^{i(-kz-\omega t)}\right]\,, \tag{19}$$

where E_0 is the amplitude, r is the reflection amplitude, k is the wave vector, and ω is the frequency.

In the metal, the wave is of the form

$$\boldsymbol{E}_m(z,t) = E_0 \hat{\mathbf{x}} t e^{i(\kappa z - \omega t)}\,, \tag{20}$$

where t is a transmission coefficient. Using Maxwell‘s equations, one can show that the electric field should satisfy the wave equation

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{v^2}\partial_t^2\right)\boldsymbol{E} = \mu\partial_t \boldsymbol{J}_f\,, \tag{21}$$

where v is the velocity of light in the material.

a) Using Maxwell‘s equations, derive the relation between the wave vector k and the frequency ω in vacuum and the relation between the wave vector κ and the frequency ω in the metal.

Solution

In a vacuum, the velocity of light $v = c$. Inserting the wave (19) into Maxwell‘s equations represented by (21), we find

$$-k^2 + \omega^2/c^2 = 0 \tag{22}$$

so that

$$k = \omega/c\,. \tag{23}$$

Similarly, in the metal, we insert the wave (20) into Maxwell‘s equations represented by (21) to find

$$-\kappa^2 + \omega^2/v^2 = -\mu\sigma i\omega \tag{24}$$

so that

$$\kappa = \pm\left(\omega^2/v^2 + i\mu\sigma\omega\right)^{1/2} \tag{25}$$

The sign of κ is determined by the condition that the wave should decay into the metal, e.g. $\mathrm{Im}[\kappa] < 0$.

b) Demonstrate that the magnetic induction in the vacuum $\boldsymbol{B}_v$ and in the metal $\boldsymbol{B}_m$ corresponding to the electric field of (19) and (20) are

$$\boldsymbol{B}_v(z,t) = E_0\frac{k}{\omega}\hat{\mathbf{y}}\left[e^{i(kz-\omega t)} - re^{i(-kz-\omega t)}\right]\,, \tag{26}$$

$$\boldsymbol{B}_m(z,t) = E_0\frac{\kappa}{\omega}\hat{\mathbf{y}} t e^{i(\kappa z - \omega t)}\,. \tag{27}$$

Solution

We first note that

$$\nabla \times (\hat{\mathbf{x}} f(z,t)) = (\hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}}\partial_y + \hat{\mathbf{z}}\partial_z) \times \hat{\mathbf{x}} f(z,t)\,, \tag{28}$$

$$= \hat{\mathbf{y}}\partial_z f(z,t)\,, \tag{29}$$

for arbitrary functions $f(z,t)$.

We use Faraday's law that relates the curl of the electric field to the rate of change of the magnetic induction (72). In the vacuum, we find

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}\,, \tag{30}$$

$$E_0 \hat{\mathbf{y}} ik \left[e^{i(kz-\omega t)} - re^{i(-kz-\omega t)}\right] = -E_0 \hat{\mathbf{y}} \frac{k}{\omega} (-i\omega) \hat{\mathbf{y}} \left[e^{i(kz-\omega t)} - re^{i(-kz-\omega t)}\right] \tag{31}$$

which is fulfilled for any E_0, k, ω, and r.

Similarly, in the metal, we have

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}\,, \tag{32}$$

$$E_0 \hat{\mathbf{y}} i\kappa t e^{i(\kappa kz-\omega t)} = -E_0 \hat{\mathbf{y}} \frac{\kappa}{\omega} (-i\omega) \hat{\mathbf{y}} t e^{i(\kappa z-\omega t)} \tag{33}$$

which is fulfilled for any E_0, k, ω, and t.

c) Determine the reflection and transmission coefficients r and t in terms of the wave vectors k, κ, the frequency ω, and, possibly, the magnetic permeability in vacuum μ_0 and the magnetic permeability in the metal μ.

Solution

In our case, the normal vector to the interface is $\hat{\mathbf{e}}_n = \hat{\mathbf{z}}$. Using the boundary condition (76) and the expressions for the electric fields of (19) and (20), we find at $z = 0$ that

$$E_0 \hat{\mathbf{z}} \times \hat{\mathbf{x}} \left[1 + r - t\right] = 0 \tag{34}$$

In other words

$$t = 1 + r\,. \tag{35}$$

Similarly, by using the boundary condition (77), the fact that there are no free surface charge currents, the expressions for the magnetic induction of (26) and (27), and the constitute relations $\boldsymbol{B}_v = \mu_0 \boldsymbol{H}_v$ and $\boldsymbol{B}_m = \mu \boldsymbol{H}_m$, we find at $z = 0$ that

$$E_0 \hat{\mathbf{z}} \times \hat{\mathbf{y}} \left[\frac{k}{\omega}\frac{1}{\mu_0}(1-r) - \frac{\kappa}{\omega}\frac{1}{\mu} t\right] = 0 \tag{36}$$

so that

$$t = \frac{k}{\kappa}\frac{\mu}{\mu_0}(1-r)\,. \tag{37}$$

Inserting Eq. (35) into Eq. (37), we solve for r and t to find

$$r = \frac{k\mu - \kappa\mu_0}{k\mu + \kappa\mu_0}\,, \tag{38}$$

$$t = \frac{2k\mu}{k\mu + \kappa\mu_0}\,. \tag{39}$$

Problem 3.

We consider a time-dependent charge density $\rho(\boldsymbol{r}, t)$ and a time-dependent charge current density $\boldsymbol{J}(\boldsymbol{r}, t)$ in a vacuum. In the Lorentz gauge

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{c^2}\partial_t V = 0\,, \tag{40}$$

the scalar potential V and the vector potential $\boldsymbol{A}$ are

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r_1 \frac{\rho(\boldsymbol{r}_1, t_r)}{|\boldsymbol{r} - \boldsymbol{r}_1|}\,, \tag{41}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int d^3r_1 \frac{\boldsymbol{J}(\boldsymbol{r}_1, t_r)}{|\boldsymbol{r} - \boldsymbol{r}_1|}\,, \tag{42}$$

where the retarded time is

$$t_r = t - |\boldsymbol{r} - \boldsymbol{r}_1|/c\,. \tag{43}$$

a) Demonstrate that the magnetic induction associated with the potentials (41) and (42) is

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int d^3r_1 \left[\frac{\boldsymbol{J}(\boldsymbol{r}_1, t_r)}{\Delta^2} + \frac{\dot{\boldsymbol{J}}(\boldsymbol{r}_1, t_r)}{c\Delta}\right] \times \boldsymbol{\Delta}\,, \tag{44}$$

where $\boldsymbol{\Delta} = \boldsymbol{r}_1 - \boldsymbol{r}$ and $\Delta = |\boldsymbol{\Delta}|$.

Solution

The magnetic induction B can be found from the vector potential A from the relation

$$\boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r}, t)\,. \tag{45}$$

We use

$$\left[\boldsymbol{\nabla} \times \frac{\boldsymbol{J}(\boldsymbol{r}_1, t_r)}{|\boldsymbol{r} - \boldsymbol{r}_1|}\right]_i = \epsilon_{ijk}\partial_j \frac{J_k(\boldsymbol{r}_1, t_r)}{|\boldsymbol{r} - \boldsymbol{r}_1|}\,, \tag{46}$$

$$= \epsilon_{ijk}(\partial_j t_r)\frac{\dot{J}_k(\boldsymbol{r}_1, t_r)}{|\boldsymbol{r} - \boldsymbol{r}_1|} + \epsilon_{ijk} J_k(\boldsymbol{r}_1, t_r)\partial_j \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_1|}\,. \tag{47}$$

$$\tag{48}$$

Next, we use

$$\partial_j t_r = -\frac{1}{c}\partial_j |\boldsymbol{r} - \boldsymbol{r}_1|\,, \tag{49}$$

$$= -\frac{\boldsymbol{\Delta}}{c\Delta^2} \tag{50}$$

and

$$\partial_j \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_1|} = -\frac{\boldsymbol{\Delta}}{\Delta^2} \tag{51}$$

to find

$$\boldsymbol{\nabla} \times \frac{\boldsymbol{J}(\boldsymbol{r}_1, t_r)}{|\boldsymbol{r} - \boldsymbol{r}_1|} = -\frac{1}{c\Delta}\boldsymbol{\Delta} \times \dot{\boldsymbol{J}}(\boldsymbol{r}_1, t_r) - \frac{1}{\Delta^2}\boldsymbol{\Delta} \times \boldsymbol{J}(\boldsymbol{r}_1, t_r) \tag{52}$$

so that we have demonstrated Eq. (44).

b) Demonstrate that the electric field associated with the potentials (41) and (42) is

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r_1 \left[\frac{\rho(\boldsymbol{r}_1, t_r)}{\Delta^2}\hat{\boldsymbol{\Delta}} + \frac{\dot{\rho}(\boldsymbol{r}_1, t_r)}{c\Delta}\hat{\boldsymbol{\Delta}} - \frac{\dot{\boldsymbol{J}}(\boldsymbol{r}_1, t_r)}{c^2\Delta}. \right], \tag{53}$$

where $\boldsymbol{\Delta} = \boldsymbol{r}_1 - \boldsymbol{r}$, $\Delta = |\boldsymbol{\Delta}|$, and $\hat{\boldsymbol{\Delta}} = \boldsymbol{\Delta}/\Delta$.

Solution

The electric field is

$$\boldsymbol{E} = -\boldsymbol{\nabla} V - \partial_t \boldsymbol{A}\,. \tag{54}$$

We compute

$$\partial_t \boldsymbol{A} = \frac{\mu_0}{4\pi} \int d^3r_1 \frac{\dot{\boldsymbol{J}}(\boldsymbol{r}_1, t_r)}{|\boldsymbol{r} - \boldsymbol{r}_1|}\,, \tag{55}$$

$$= \frac{1}{4\pi\epsilon_0} \int d^3r_1 \frac{\dot{\boldsymbol{J}}(\boldsymbol{r}_1, t_r)}{c^2|\boldsymbol{r} - \boldsymbol{r}_1|}\,, \tag{56}$$

where we have used that $1/c^2 = \mu_0\epsilon_0$.

We also compute

$$\boldsymbol{\nabla} V = \frac{1}{4\pi\epsilon_0} \int d^3r_1 \boldsymbol{\nabla} \frac{\rho(\boldsymbol{r}_1, t_r)}{|\boldsymbol{r} - \boldsymbol{r}_1|} \tag{57}$$

$$= \frac{1}{4\pi\epsilon_0} \int d^3r_1 \left[-\rho(\boldsymbol{r}_1, t_R)\frac{\hat{\boldsymbol{\Delta}}}{\Delta^2} - \frac{1}{c}\dot{\rho}(\boldsymbol{r}_1, t_r)\frac{\hat{\boldsymbol{\Delta}}}{\Delta} \right] \tag{58}$$

Inserting Eqs. (56) and (58) into Eq. (54), we find Eq. (53).

c) We consider a dipole configuration. A charge $q(t)$ is at point $\boldsymbol{d}_0/2$, and a charge $-q(t)$ is at point $-\boldsymbol{d}_0/2$. We use complex notation and assume that there is a harmonic time dependence of the charge

$$q(t) = q_o e^{-i\omega t}\,. \tag{59}$$

Demonstrate that the scalar potential is (in the Lorentz gauge)

$$V(\boldsymbol{r}, t) = \frac{q_0}{4\pi\epsilon_0} e^{-i\omega t} \left[\frac{e^{i\omega r_+/c}}{r_+} - \frac{e^{i\omega r_-/c}}{r_-} \right], \tag{60}$$

where

$$r_\pm = |\boldsymbol{r} \mp \boldsymbol{d}_0/2| = r\sqrt{1 \mp \boldsymbol{r}\cdot\boldsymbol{d}_0 + (\boldsymbol{d}_0/2)^2}\,. \tag{61}$$

Solution

We use Eq. (41) to find

$$V(\boldsymbol{r},t) = \frac{q_0}{4\pi\epsilon_0}\int dr_1^3 \frac{e^{-i\omega(t-|\boldsymbol{r}-\boldsymbol{r}_1|/c}}{|\boldsymbol{r}-\boldsymbol{r}_1|}\left[\delta(\boldsymbol{r}_1-\boldsymbol{d}_0/2)-\delta(\boldsymbol{r}_1+\boldsymbol{d}_0/2)\right]\,, \tag{62}$$

$$= \frac{q_0 e^{-i\omega t}}{4\pi\epsilon_0}\left[\frac{e^{i\omega r_+/c}}{r_+} - \frac{e^{i\omega r_-/c}}{r_-}\right] \tag{63}$$

as we should demonstrate.

d) Compute the scalar potential $V(\boldsymbol{r},t)$ in the radiation zone when the distance $r = |\boldsymbol{r}|$ is much larger than the separation between the charges $d_0 = |\boldsymbol{d}_0|$ and when the distance r is much larger than the wavelength $\lambda = 2\pi/k$, where $k = \omega/c$.

Solution

In the radiation zone, we expand to the lowest order the expressions for the distances r_+ and r_-:

$$r_\pm \approx r\left(1 \mp \hat{\mathbf{r}}\cdot\boldsymbol{d}_0/2r\right) \tag{64}$$

so that

$$\frac{1}{r_\pm} \approx \frac{1}{r}\left(1 \pm \hat{\mathbf{r}}\cdot\boldsymbol{d}_0/2r\right)\,. \tag{65}$$

We also use

$$e^{ikr_\pm} \approx e^{ikr}e^{\mp ik\boldsymbol{r}\cdot\boldsymbol{d}_0/2r^2} \approx e^{ikr}\left(1 \mp ik\boldsymbol{r}\cdot\boldsymbol{d}_0/2r^2\right)\,. \tag{66}$$

We then find

$$V(\boldsymbol{r},t) \approx \frac{q_0}{4\pi\epsilon_0}\frac{e^{i(kr-\omega t)}}{r}\left[\left(1 - ik\boldsymbol{r}\cdot\boldsymbol{d}_0/2r\right)\left(1+\boldsymbol{r}\cdot\boldsymbol{d}_0/2r^2\right) - \left(1+ik\boldsymbol{r}\cdot\boldsymbol{d}_0/2r\right)\left(1-\boldsymbol{r}\cdot\boldsymbol{d}_0/2r^2\right)\right]\,, \tag{67}$$

$$\approx \frac{q_0}{4\pi\epsilon_0}\frac{e^{i(kr-\omega t)}}{r}\left[-ik\boldsymbol{r}\cdot\boldsymbol{d}_0/r + \boldsymbol{r}\cdot\boldsymbol{d}_0/r^2\right] \tag{68}$$

In the radiation zone, the second term can be neglected, and we find

$$V(\boldsymbol{r},t) \approx \frac{q_0}{4\pi\epsilon_0}\frac{e^{i(kr-\omega t)}}{r}\left[-ik\boldsymbol{r}\cdot\boldsymbol{d}_0/r\right]\,. \tag{69}$$

A Maxwell‘s Equations

Maxwell‘s equation in vacuum for the electric field $\boldsymbol{E}$, the displacement field $\boldsymbol{D}$, the magnetic induction $\boldsymbol{B}$, and the magnetic field $\boldsymbol{H}$ are

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_f \,, \tag{70}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \,, \tag{71}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \,, \tag{72}$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial D}{\partial t} \,, \tag{73}$$

in terms of the free charge density ρ_f and the free charge current density $\boldsymbol{J}_f$.

B Constitute Relations

In linear and isotropic media, we have

$$\boldsymbol{D} = \epsilon \boldsymbol{E} \,, \tag{74}$$

$$\boldsymbol{B} = \mu H \,, \tag{75}$$

where ϵ is the dielectric constant and μ is the magnetic permeability.

C Boundary Conditions for Electromagnetic Fields

At interfaces between material 1 and material 2, the boundary conditions are

$$\hat{\mathbf{e}}_n \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0 \,, \tag{76}$$

$$\hat{\mathbf{e}}_n \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{K}_s \,, \tag{77}$$

$$\hat{\mathbf{e}}_n \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = \sigma_S \,, \tag{78}$$

$$\hat{\mathbf{e}}_n \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0 \,, \tag{79}$$

where $\hat{\mathbf{e}}_n$ is a unit vector normal to the interface, σ_S is the surface charge density, and $\boldsymbol{K}_s$ is the surface charge current density.

D Spherical Coordinates

In spherical coordinates r, θ, and ϕ, the gradient is

$$\nabla t = \hat{\mathbf{r}} \frac{\partial t}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial t}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{r \sin\theta} \frac{\partial t}{\partial \phi} \,. \tag{80}$$

The divergence is

$$\nabla \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\,. \tag{81}$$

The Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,. \tag{82}$$

E Products of matrices

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})\ , \tag{83}$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\,(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}\,(\boldsymbol{A}\cdot\boldsymbol{B})\ . \tag{84}$$

F Integral Theorems

The divergence theorem is

$$\int d^3r\nabla\cdot\boldsymbol{A} = \oint \boldsymbol{A}\cdot d\boldsymbol{S}\,. \tag{85}$$

Stoke‘s theorem (or the curl theorem) is

$$\int(\nabla\times\boldsymbol{A})\cdot d\boldsymbol{S} = \oint \boldsymbol{A}\cdot d\boldsymbol{l}\,. \tag{86}$$

G Some Useful Results

We define $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_1$, $R = |\boldsymbol{R}|$, and $\hat{\mathbf{R}} = \boldsymbol{R}/R$. Then

$$\boldsymbol{\nabla}\cdot\frac{\hat{\mathbf{R}}}{R^2} = 4\pi\delta(\boldsymbol{R})\,, \tag{87}$$

$$\boldsymbol{\nabla}\frac{1}{R} = -\frac{\hat{\mathbf{R}}}{R^2}\,, \tag{88}$$

$$\boldsymbol{\nabla}^2\frac{1}{R} = -4\pi\delta(\boldsymbol{R})\,. \tag{89}$$