NTNU – Trondheim
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Department of Physics

Examination paper for TFY4245 Faststoff-fysikk, videregående kurs

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Permitted examination support material:

Alternative C:

Standard pocket calculator

K. Rottmann: Matematical Formulae (all language editions)

S. Barnett & T.M. Cronin: Mathematical Formulae

Other information:

Language: English

Number of pages: 5 (incl. front page and attachments).

Checked by:

Date Signature

Some relationships that may be found useful:

Maxwell equations:

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \cdot \vec{D} = \rho$$

Electromagnetic wave equation:

$$\nabla^2 \vec{E} = \mu_0 \frac{\partial^2 \vec{D}}{\partial t^2}$$

Electromagnetic identities:

$$\vec{D} = \varepsilon_0 \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\vec{B} = \mu_0 (\vec{M} + \vec{H})$$

Curie-Brillouin relationship:

$$M = \frac{N}{V} \cdot g(JLS) J \mu_B \cdot B_J \left(\frac{g(JLS) J \mu_B \mu_0 H}{k_B T} \right)$$

with: $$B_J(x) = \frac{2J+1}{2J} \coth\left(\frac{2J+1}{2J} x \right) - \frac{1}{2J} \coth\left(\frac{x}{2J} \right)$$

$$g(JLS) = \frac{3}{2} + \frac{1}{2} \frac{S(S+1) - L(L+1)}{J(J+1)}$$

Curies law:

$$\chi = \frac{C}{T}$$

Problem 1 – plasma excitation (33.33 %).
The response of the free electrons in a metal to external electromagnetic fields can be described by a cold plasma model where the charges of the free electrons are balanced by those of the (immobile) ion cores.

Assume that an electromagnetic plane wave is impinging on the sample, with components on the form $\vec{\theta}(\vec{r},t)=\vec{\theta}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, where θ_0 symbolizes the field amplitude, k is the wave number and ω is the radian frequency. Assume the plasma to be non-magnetic ($\mu=\mu_0$)

a) Let $\vec{r}$ be the position for a single free electron. Set up an equation of motion for the electron due to the forces exerted on it by the external electromagnetic field, taking into account also resistive losses.

Show that in the long wavelength limit, the collective response of the free electron system can be given as:

$$\varepsilon_{el}(\omega)=1-\frac{ne^2}{\varepsilon_r\varepsilon_0 m_e}\frac{\tau^2}{\omega^2\tau^2+i\omega\tau}=1-\frac{\omega_p^2\tau^2}{\omega^2\tau^2+i\omega\tau}$$

where n is the number density of electrons, τ the relaxation time associated with resistive losses, e and m_e the charge and the rest mass of the electron, respectively, ε_0 the vacuum permittivity, and ε_r the dielectric constant of the positive ion core medium.

In the rest of the problem we restrict ourselves to combinations of temperatures and electromagnetic radiation where $\omega\tau >> 1$, such that resistive losses can be neglected.

b) Find a dispersion relation for the electromagnetic wave in the plasma medium in the long wavelength limit. Sketch the resulting $\omega(k)$ in a figure, and account explicitly for the excitation of the plasma at $\omega = \omega_p$.

A metal sample like the one above (S1) is in the form of a thin slab. We place S1 (z>0) in direct contact with a slab of another metal, S2 (z<0), such that the interface between the samples is located in z = 0.

S1 has plasma frequency ω_{p1} while S2 has plasma frequency $\omega_{p2.}$ When brought together, solutions of the Poisson equation for the two metal slabs yield potentials $\varphi_1(x,z)=A\cos(kx)e^{-kz}$ (z > 0) and $\varphi_2(x,z)=A\cos(kx)e^{kz}$ (z <0), for which the associated electric fields are given by $\vec{E}=-\nabla\varphi$. The boundary conditions that apply between the two media, require both the tangential component of the electric field and the normal component of the dielectric field to be continuous at the interface, i.e. $E_{1x}= E_{2x}$ and $D_{1z}= D_{2z}$ in z =0.

c) Show that the characteristic long wavelength-limit plasmon frequency at the interface is given by

$$\omega=\left[\frac{1}{2}(\omega_{p1}^2+\omega_{p2}^2)\right]^{1/2}$$

In this problem, (and in the lectures/curriculum), we have employed a free electron gas (Drude model) when deriving the plasmon behavior. A more realistic description of the free electrons in a solid is given by a so-called nearly-free electron model, taking into account the periodic potential via the Bloch-wave formalism. How would the introduction of a nearly-free electron model affect our long wavelength limit plasma model?

Problem 2. Super conductivity (33,33 %)

a) Sketch and explain the magnetization curves (i.e. magnetization M vs magnetic field H) for type I and type II superconductors, respectively.

Employing Ginzburg-Landau wave mechanics, the superconducting current density for Cooper pairs may be written

$$\vec{j}_{c.p.}(r) = -\frac{e}{m_e}\, n_{c.p.}(\hbar\nabla\theta(r) + 2e\vec{A})$$

where the cooper pairs are described by a wave function $\psi(r) = \sqrt{n_{c.p.}}e^{i\theta(r)}$ with $n_{c.p.}$ as the cooper pair concentration, and $\theta(r)$ as a phase. $\vec{A}$ is the vector potential related to the magnetic field such that $\nabla\times\vec{A} = \vec{B}$

b) Show that the London equation, $\nabla^2\vec{B} = \lambda_L^{-2}B$, can be derived from the expression for superconducting current density in combination with Maxwell's equations, where λ_L is a characteristic length.

Assume sample in the shape of a long straight cylinder with radius R, carrying a steady superconducting current, $I_{s.c}$, along the cylinder axis, $\vec{l}$. The cylinder is a type I superconductor. Find $B(r)$, where r is the radial distance from $\vec{l}$ located in $r = 0$ in the center of the cylinder. Sketch the result in a graph, and discuss the physical meaning of λ_L.

c) In the Ginzburg-Landau theromodynamic theory for superconductivity, the free energy function for a bulk superconductor of type I, may be expressed as

$$g(|\psi|,T) = g_0 + \gamma(T-T_C)|\psi|^2 + g_4|\psi|^4 \qquad (1)$$

where g_0, γ and g_4 are positive constants, while $|\psi| = n_{c.p}^{1/2}$ serves as the order parameter.

Find an expression for the equilibrium value of the cooper pair concentration for $T < T_C$.

Type II superconductors may involve spatial variations in $n_{c.p}(r)$. Thus, eqn. (1) does not hold since it involves a spatially homogeneous order parameter. Neglecting all field effects, it can be shown that the Ginzburg-Landau free energy function, at any given temperature $T < T_C$, and in regions with a spatially varying order parameter may be expressed

$$g(\psi(\mathrm{r})) - g_0 \propto -\frac{|\psi|^2(r)}{|\psi|_{eq}^2} + \frac{|\psi|^4(r)}{2|\psi|_{eq}^4} + c_{GL}^2\frac{\nabla^2|\psi|^2(r)}{|\psi|_{eq}^2}$$

where $|\psi|_{eq}$ is the homogeneous equilibrium bulk concentration for $T < T_C$ as determined from eqn. (1). Determine the dimension and temperature dependence of the parameter c_{GL}, and discuss its physical meaning.

Problem 3 Magnetism (33.33%).

Crystals of Cr(III)Br_3 exhibit ferromagnetic properties at temperatures $T < T_C = 32.7$ K, and paramagnetic response for $T > T_C$. The magnetic moments arise from the Cr^{3+} ions. These occupy nearest neighbour lattice sites of an equidistant spacing, and their magnetic moments interact with one another through a direct exchange. The crystal symmetry of Cr(III)Br_3 is such that ferromagnetic domains align parallel or antiparallel with the c-axis.

In a generalised microscopic model, containing both Heisenberg exchange terms and the spin interaction with an external magnetic field, $\vec{H}_{ext}$, the energy associated with magnetic response may be expressed

$$U = -\sum_{i \neq j} \Im(\Delta\vec{r}_{ij})\vec{S}_i \cdot \vec{S}_j + \sum_i g(JLS)\mu_B\mu_0\vec{H}_{ext} \cdot \vec{S}_i = \sum_i g(JLS)\mu_B\mu_0\vec{H}_{eff} \cdot \vec{S}_i \qquad (2)$$

The first sum is taken over all exchanging spin pairs ij. $\Im$ represents the exchange energy and its value depends on the interatomic distance Δr_{ij}. For simple systems in equilibrium, all exchange pair distances, Δr_{ij}, may be assumed identical, and thus $\Im = <\Im>_T$ is constant.

a) Cr^{3+} has electronic configuration $3d^3$ in the ground state. Determine the effective number of Bohr magnetons, $p = g(JLS)[J(J+1)]^{1/2}$ for free Cr^{3+} ions.

In crystals of Cr(III)Br_3, measurements show a paramagnetic yield which corresponds to p for Cr^{3+} being approximately 5 times larger than the value calculated (correctly) above. Give a brief account for reasons that could cause the discrepancy.

b) In the direct exchange that applies to Cr(III)Br_3, $\Im$ in eqn. (1) is non-zero only for the two nearest neighbour spins along $\vec{c}$. Employ a mean field approach to the microscopic model presented in eqn. (2) to show that this yields the standard Weiss molecular mean field $\vec{H}_{eff} = \vec{H}_{ext} + \lambda\vec{M}$, and express the constant λ by quantities defined in eqn. (2). (Hint: Express $\vec{H}_{eff}$ using eqn. (2), and replace microscopic quantities by their thermal average values under the assumption that the net magnetisation of the system is accounted for by N equivalent spins).

Use the Weiss molecular mean-field representation to find an expression for the paramagnetic susceptibility for $T > T_C$ in the presence of an external magnetic field H_{ext}, and show that $T_C = 2S(S+1) < \Im >_{T_C} (3k_B)^{-1}$.

c) Express $M(T)$ in the ferromagnetic region in the absence of external magnetic fields, based both on microscopic and mean field quantities. Compare the expressions and discuss possible discrepancies between the two approaches.