Some relationships that *may* be found useful.

Maxwell equations:

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \cdot \vec{D} = \rho$$

Electromagnetic wave equation:

$$\nabla^2 \vec{E} = \mu_0 \frac{\partial^2 \vec{D}}{\partial t^2}$$

Electromagnetic identities:

$$\vec{D} = \epsilon_0 \epsilon \vec{E} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1+\chi)\vec{E} \qquad \vec{B} = \mu_0 \mu \vec{H} = \mu_0(\vec{M} + \vec{H}) = \mu_0(1+\chi)\vec{H}$$

Curie-Brillouin relationship:

$$M = ng(JLS)\mu_B J B_J \left(\frac{g(JLS) J \mu_0 \mu_B H}{k_B T} \right)$$

with:

$$B_J(x) = \frac{2J+1}{2J} \coth\left(\frac{2J+1}{2J} x \right) - \frac{1}{2J} \coth\left(\frac{x}{2J} \right)$$

$$g(JLS) = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}$$

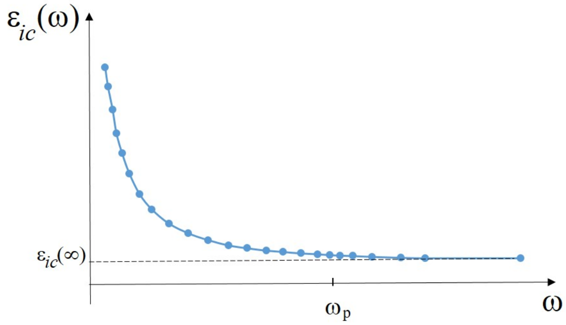
Curies law:

$$\chi = \frac{C}{T}$$

Some physical constants:
electron mass: $m_e = 9.109 \cdot 10^{-31}$ kg
elementary charge: $e = 1.602 \cdot 10^{-19}$ C
vacuum permittivity: $\epsilon_0 = 8.8542 \cdot 10^{-}12$ F/m
vacuum permeability: $\mu_0 = 4\pi \cdot 10^{-7}$ H/m
Boltzmann's constant: $k_B = 1.38065 \cdot 10^{-23}$J/K
Bohr magneton: $\mu_B = e\hbar/2m_e = 9.274 \cdot 10^{-24}$J/T

Problem 1 - Plasmons
a) In the long wavelenght limit, derive the expression $\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \omega_p^2 = \frac{ne^2}{m_e\epsilon_0}$ as the mean-field dielectric response of a free electron gas with number density n, to an electric field $\vec{E} = \vec{E}_0 \exp(i(\vec{k}\vec{r} - \omega t))$.

In a metal, the free electron gas and the remaining ion cores constitute a plasma. The figure below illustrates a possible dielectric response function for the ion cores, $\epsilon_{ic}(\omega)$, over a frequency range from infrared to hard UV/soft X-rays, well above ω_p.



Explain the frequency dependence in ϵ_{ic}. Derive an expression for the dielectric response of the whole plasma in the long wavelenght limit, valid within the range where plasmons may be exited.

b) Show clearly that metals are opaque (non-transparent) for light with $\omega < \bar{\omega}_p$. You can assume the metal to be non-magnetic, i.e. $\mu(\omega) \equiv 1$, so that the permeability is as for vacuum.

Na is an alkali metal with valency 1, (i.e. one free electron per atom). Na crystallises in a body centred cubic cell, with lattice parameter $a = 0.429$ nm. Calculate the wavelength cutoff for a Na metal when you assume $\epsilon_{ic}(\infty) = 1.05$.

Problem 2 - Polarisation
Consider a solid consisting of independent dipoles that are able to rotate freely, and are randomly oriented in the absence of external fields. The potential energy of a dipole of moment $\vec{p}$ in an applied electric field $\vec{E}$ is

$$V = -\vec{p} \cdot \vec{E} = -pE\cos\theta.$$

The ensemble of dipoles can be treated via statistical mechanics by employing a Boltzman distribution of the kind $f(\theta) = e^{-V/k_BT}$, to express the probability of finding a dipole in orientation θ under the influence of en external field $\vec{E}$, at temperature T.

a)Show that the total polarisation for a system of volume V, consisting of N such independent rotational dipoles of the same magnitude p, can be expressed

$$P = \frac{nk_BT}{E}\left[\frac{pE}{k_BT}\coth\left(\frac{pE}{k_BT}\right) - 1\right].$$

(Hint: The ensemble of randomly oriented dipoles with fixed magnitudes may be represented as a continuum by a spherical surface with radius p).

The orientational polarisability per dipole can be given as $\alpha = \frac{P}{En}$. Show that for weak fields, $\alpha \propto T^{-1}$.

b) If the electric field oscillates, the dipoles, following the field, will flip back and forth as the field reverses its direction during each cycle. However, the dipoles may experience "friction", due to their interaction with other molecules/dipoles in the system, causing some loss of energy, also known as dielectric loss. This means there is a charcteristic relaxation time, τ, involved, as well as a possible phase lag between the field and the polarisation. Thus, the dielectric function, $\epsilon = \epsilon_0 + \alpha n = \epsilon_1 + i\epsilon_2$ is a complex function (n is the number density of dipoles).

Sketch graphs of ϵ_1 and ϵ_2 against $\log \omega\tau$, where ω is the oscillation frequency of the external electric field, and give explicit reasons for the appearances you have suggested for the real and imaginary dielectric responses.

Problem 3 - Superconductivity

The Ginzburg-Landau theory is based on a free energy density which may be expressed:

$$f_{sc}(T,\psi) = f_n(T) + \gamma(T - T_c)\,|\psi(\vec{r})|^2 + \frac{1}{2}\beta(T)\,|\psi(\vec{r})|^4 + \frac{1}{2m_e}\,|\vec{p_{op}}\psi(\vec{r})|^2\,,$$

where f_{sc} and f_n are the free energy densities of the superconducting and normal state, respectively, while T_c denotes the temperature of the normal-supercondcting phase transition. $\psi(\vec{r}) = |n_{cp}(\vec{r})|^{1/2}\,e^{i\theta(\vec{r})}$ represents a complex order parameter with $n_{cp}(\vec{r})$ as the number density of cooper-pairs. $\vec{p}_{op} = -i\hbar\nabla + 2e\vec{A}$ is a generalised momentum operator where $\vec{A}$ is the vector potential, related to the magnetic field through $\nabla \times \vec{A} = \vec{B}$. γ is a positive constant, and $\beta(T)$ a function which varies smoothly with temperature. Close to $T_c, \beta(T) \simeq \beta$ =constant.

a) Find the equilibirum value $(n_{cp})_{eq}$ for $T < T_c$, but close to T_c, for the bulk of a type I superconductor, and determine the temperature dependent critical field $\vec{H}_c(T)$.

The superconductive current density formed by the Cooper pairs can be expressed $\vec{j}_{cp} = \frac{q}{2m}\left[\psi^\star \vec{p}_{op}\psi + \psi(\vec{p}_{op}\psi)^\star\right]$, with q and m as the Cooper pair charge and mass, respectively.

b) Assume a type I superconductor in thermal equilibrium. Derive the London equation, $\nabla^2\vec{B} = \vec{B}/\lambda_L^2$ by combining the expression for the supercoductive current density with Maxwells equations. What is the physical interpretation of $\lambda_L(T)$?

c) Express the equilibirum value for the order parameter from the Ginzburg Landau free energy density function in a transition region between the bulk superconductive state and the so-called vortex region in a type II superconductor, and use the result to find an explicit form for the so-called coherence length, $\xi(T)$.

Problem 4 - Magnetism

a) Apply Hund's rules to determine the ground state of Gd^{3+} with electron configuration $4f^7 5s^2 p^6$ and V^{2+} with configuration $3d^2$. Express your answers in terms of J, L, S quantum numbers for the partially filled shells.

b) We compose solids of the ions in a) and find them to exhibit paramagnetism at room temperature. Under the assumption of weak magnetic fields, show that Curies constant may be expressed $C = \frac{np^2 \mu_B^2 \mu_0}{3k_B}$, where $p = g(JLS)(J(J+1))^{1/2}$ is the effective number of Bohr magnetons.

In the two solids, the density of Gd^{3+}-ions is $n = 2.62 \cdot 10^{+28} \mathrm{m}^{-3}$, while the density of V^{2+}-ions is $n = 3.59 \cdot 10^{+28} \mathrm{m}^{-3}$. Calculate the Curie constants for the two paramagnetic solids.

c)The Gd^{3+} solid is a metal which happens to undergo a paramagnetic-ferromagnetic phase transition at 289 K. Employ a Weiss molecular mean-field direct exchange, $\vec{H}_{eff} = \vec{H}_{ext} + \lambda \vec{M}$ to find expressions for the magnetisation of the Gd^{3+} crystal above and below T_c.

What is the ratio $\chi(300\mathrm{K})/\chi(290\mathrm{K})$ for the paramagnetic susceptibility of Gd^{3+}?

Determine the value of λ.

d) V^{2+} is also metallic, and remains paramagnetic until ~ 5.27 K, where it undergoes a transition from a normal to a superconducting state.

Let each Vanadium ion contribute with a magnetic moment $\vec{\mu}$, such that the energy levels of the system in an external magentic field are $E_{J_z} = -\vec{\mu} \cdot \vec{B} = J_z g(JLS) \mu_B B$, where J_z are the asimuthal quantum numbers, with values $J, (J-1), ... - J$.

For simplicity, neglect the normal-superconductive transition, and assume that the system remains paramagnetic all the way until 0 K. Assume the system to be in equilibirum, and employ a Boltzman distribution on the form $f = e^{-E_{J_z}/k_B T}$ to determine the relative popoluations of each ionic state, i.e. N_{J_z}/N, where $N = \sum_{-J}^{J} N_{J_z}$.

The magentisation of the system can be given as $M = \sum_{-J}^{J} M_{J_z} N_{J_Z}$, where $M_{J_z} = -\frac{1}{V} \frac{\partial E_{J_Z}}{\partial B}$. Dedudce the total magnetisation of the system using these relationships and the relative populations from above, and show that this can be expressed:

$$M = \left[\frac{4 \sinh(2\mu_B B / 2k_B T)}{2 \cosh(2\mu_B B / 2k_B T) + 1} \right] n \mu_B.$$

Give a *brief* account for how the normal-superconductive transition will affect the magentisation and the paramagnetic response in V^{+2} when you appraoch the transition temperature from above? Justify your answer.