Table – Answers to Problem 4 questions

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
C	B	A	D	D	A	C	D	C	C	B	A	B	C	A	D(ABC)

Problem 1.

a) In second-order transitions, the quantity associated with the ordering increases continuously from $\langle\eta\rangle = 0$ at the phase transition, where the phases of the ordered and disordered state are in equilibirum. => Close to T_C, $G(\eta)$ is described mainly by the lower-order terms.

Another requirement to the ordered phase of second order phase transitions is $G(\eta) = G(-\eta)$, which may hold in general only if $g_1=g_3=0$.

At the phase transition temperature the disordered and the ordered phase must be in equilibrium

$$\Rightarrow \; G(\eta) = g_0 + \tfrac{1}{2} g_2\eta^2 + \tfrac{1}{4} g_4\eta^4$$

System in equilibrium at any T:

$$\frac{\partial G}{\partial \eta} = 0 = \eta(g_2 + g_4\eta^2)$$

$$): \eta = 0 \quad \vee \quad \eta = \pm\sqrt{-\frac{g_2}{g_4}}$$

1. T > T

Here, in general $g_0 = G_1(T)$, which may differ from $G_1(T_C)$, but close to T_C we assume this can be negelcted.

$g_2 > 0$ => $g_4 < 0$ gives real solution, while $g_4 > 0$ gives imaginary solution.

Energy minima:

i. $\eta = 0 \Rightarrow G = g_0$ (T)

ii. For the real non-zero solution, $G(\eta) = g_0 + \dfrac{1}{4}\dfrac{g_2^2}{|g_4|} > g_0$

1. $T < T_C$:

$g_2 < 0$ => $g_4 > 0$ gives real solution, while $g_4 < 0$ gives imaginary solution.

Energy minima:

i. $\eta = 0 \Rightarrow G = g_0$

ii. For the real non-zero solution, $G(\eta) = g_0 - \frac{1}{4}\frac{g_2^2}{g_4} < g_0$

Thus, $g_4 = \frac{2\gamma}{\eta^2} |\Delta T| \;\; > 0$ is consistent with:

Minimum G(η) = g_0 and η = 0 for T > T_C

and

Mimumum $G(\eta) = g_0 - \frac{1}{4}\frac{g_2^2}{g_4} < g_0$ and $\eta = \pm\sqrt{-\frac{g_2}{g_4}}$ for T < T_C

b) With the external field present, the Gibbs free energy becomes

$$G(\eta) - \vec{\theta}_{ext}\vec{\eta} = g_0 + \tfrac{1}{2} g_2\eta^2 + \tfrac{1}{4} g_4\eta^4 - \vec{\theta}_{ext}\vec{\eta}$$

where the vectors in the last term can be dropped when the external field is considered to be aligned parallel or antiparallel with the ordering parameter.

System in thermal equilibrium at any temperature,

$$\frac{\partial G}{\partial \eta} = 0 = -\theta + g_2\eta + g_4\eta^3$$

$$) : \theta = g_2\eta + g_4\eta^3$$

For a linear system, the susceptibility

$$\chi = \left[\frac{d\theta}{d\eta}\right]^{-1}_{\theta=0} = (g_2 + 3g_4\eta_0^2)^{-1}$$

where η_0 is the equilibrium value of the order parameter in absence of external fields.

Thus, for $T > T_C$ where $\eta_0 = 0$

$$\chi = 1/g_2 = \frac{1}{\gamma(T - T_C)}$$

and for $T < T_C$ where $\eta_0 = \pm\sqrt{-\frac{g_2}{g_4}}$

$$\chi = \frac{1}{g_2 + 3g_4(\frac{-g_2}{g_4})} = \frac{1}{-2g_2} = \frac{1}{2\gamma(T_C - T)}$$

c) The Hysteresis behaviour relate overall sample response to an external field, which includes both the (microscopic) susceptibility as

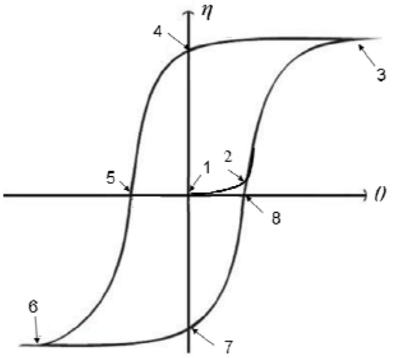

Figure 1. Hysteresis loop

well as reorientation of domains. In its current form, eqn. (1) only accounts for the free energy of a single domain.

$V_{\uparrow\eta}$: Total volume of all domains with $\eta > 0$

$V_{\downarrow\eta}$: Total volume of all domains with $\eta < 0$

$\uparrow\eta$: The absolute mean-field respons, per unit volume, of the order parameter in orientation parallel to the external field.

$\downarrow\eta$: The absolute mean-field respons, per unit volume, of the order parameter in orientation anti-parallel to the external field.

Points of the graph:
1: $V_{\uparrow\eta} \approx V_{\downarrow\eta}$; $\eta\uparrow = \eta\downarrow$ => no net contrib. to the macroscopic sample

2: No reorientation of domains up to this point, i.e. $V_{\uparrow\eta} \approx V_{\downarrow\eta}$, but the absolute mean-field responses may be slightly unbalanced $\eta\uparrow > \eta\downarrow$. From this point on, the main contribution to η by reorienting domains.

3: $\eta \to \eta_S$, the saturation limit, the sample has reached a monodomain state, i.e. the pooled state, and further increase in θ is not anymore causing a response in the system.

4: $\eta \to \eta_R$, the remnant η. The external field has returned to zero, but η keeps a remant value even in the absence of a field. Over time this will relax, but may be quite persistent in some systems.

5: $\theta \to -\theta_C$, the (negative) coercieve field. Here η is returned back to 0, where the situation is roughly the same as in 1, but with a non-zero field. The coercieve field is the minimum field required to reverse the net ordering of the sample.

6: $\eta \to -\eta_S$, the negative saturation limit, as point 3: but η pooled in the other direction.

7: $\eta \to -\eta_R$, the negative remnant state of the order param. As in point 4, but with opposite sign of η

8: $\theta \to \theta_C$, the coercieve field. $\eta = 0$, as in point 5.

Problem 2

a) Eqn. of motion:

$$m\frac{d\vec{v}}{dt} = -e\vec{E} \qquad (2)$$

With $\rho = 0$, all relaxation times, $\tau \to \infty$, i.e. dissipative terms vanish.

Introduce current density, $\vec{j} = -ne\vec{v}$, in eqn. (2)

$$\frac{d\vec{j}}{dt} = -\frac{e^2 n}{m}\vec{E}$$

Maxwell:

$$\frac{m}{e^2 n}\frac{\partial}{\partial t}\left(\nabla\times\vec{j}\right)=\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

=> $\nabla\times\vec{j}+\frac{e^2 n}{m}\vec{B}=0$ q.e.d.

b) We use:

$$\nabla\times\vec{H}=\vec{j}+\frac{\partial\vec{D}}{\partial t}\quad\text{and}\quad\vec{B}=\mu_0(\vec{H}+\vec{M})$$

$$\Rightarrow\nabla\times\vec{B}=\mu_0\vec{j}+\mu_0\frac{\partial\vec{D}}{\partial t}+\mu_0\nabla\times\vec{M}$$

No magnetisation currents and no time-dependent displacement currents:

$$\nabla\times\vec{B}=\mu_0\vec{j}$$

Proceed as when deriving the wave eqn.:

$$\nabla\times\nabla\times\vec{B}=\nabla(\nabla\cdot\vec{B})-\nabla^2\vec{B}=-\nabla^2\vec{B}=\mu_0\nabla\times\vec{j}$$

where a vector identity is used for the double curl on the lhs, and then Maxwell for the divergence of B. Introducing the identity given on the rhs to the London eqn. give us the first relation we seek, i.e.

$$\mu_0\nabla\times\vec{j}+\frac{\mu_0 e^2 n}{m}\vec{B}=-\nabla^2\vec{B}+\frac{\mu_0\varepsilon_0 ne^2}{m\varepsilon_0}\vec{B}$$

$$\Rightarrow\quad\nabla^2\vec{B}=\frac{ne^2}{m\varepsilon_0 c^2}\vec{B}$$

and for the current density:

$$\nabla\times\nabla\times\vec{j}=\nabla(\nabla\cdot\vec{j})-\nabla^2\vec{j}$$

The first term on the rhs is zero when we consider a steady DC current. Thus, we have:

$$\frac{ne^2}{m}\nabla\times\vec{B}=-\nabla\times\nabla\times\vec{j}=\nabla^2\vec{j}\quad\vee\quad\nabla\times\vec{B}=\mu_0\vec{j}$$

$$\Rightarrow\nabla^2\vec{j}=\frac{ne^2}{m\varepsilon_0 c^2}\vec{j}$$

c) Outside S.C, we have

$$\vec{B}=B_{ext}\cdot\hat{k}$$

And deep inside S.C, $B = 0$.

We use the London eqn. to determine the behaviour of B as a function of x, starting from the S.C.-vacuum boundary in x = 0, i.e.

$$\frac{d^2B(x)}{dx^2} = \frac{1}{\lambda_L^2}B(x)$$

A general solution to the problem on the form

$$B(x) = Ae^{-x/\lambda_L} + Be^{x/\lambda_L}$$

We can conclude automatically that B=0 (otherwise B will increase exponentially as we move deep into the S.C).

Second, we can determine A by requiring B(x) to be continuous at the S.C-vacuum interface, i.e. $A=B(0)=B_{\text{ext}}$, yielding

$$B(x) = B_{ext}e^{-x/\lambda_L}$$

The same procedure can be followed to find a solution for the current density.

The physical interpretation of the situation: λ_L is the so-called London penetration depth, which measures how deep the exterior magnetic field penetrates into a superconductor. For type-II superconductors it also corresponds to the spatial extention of vortices, perpendicular to the magnetic flux lines. Both in type I and type II superconductors, λ_L also gives the spatial extention of the superconducting current density.

The London equation explains the so-called Meissner effect, and is important in a wider physical context since it tells us how we may have the Meissner-effect without violating fundamental relationships of physics, like e.g. Amperes law. It shows that superconducting currents are confined within regions close to the surfaces of type I S.C.s, and in type II S.C.s also adjacent to vortices.

Problem 3

a) $\Im < 0$ makes the antiparallel nearest-neighbour exchange energetically favourable.

With U_0 relating to a ground state with all neighbour spins anti-parallel, the reversal of one spin , e.g. spin no. p reversed from $\uparrow$ to $\downarrow$ give us:

$$\begin{aligned} U &= U_0 - (-2\Im \uparrow_p \downarrow_{p+1}) - (-2\Im \downarrow_{p-1} \uparrow_p) + (-2\Im \downarrow_p \downarrow_{p+1}) + (-2\Im \downarrow_{p-1} \downarrow_p) \\ &= U_0 - (-2|\Im|S^2) - (-2|\Im|S^2) + (2|\Im|S^2) + (2|\Im|S^2) \\ &= U_0 + 8|\Im|S^2 \end{aligned}$$

On the right hand side, the first to terms are the spin no. p containing terms extracted from U_0, while the two last terms are from additing the p containing terms back again, after spin p has been flipped.

In result, the energy of the system is increased by $8|\Im|S^2 = 2|\Im|$ when all spins are $S = \pm\frac{1}{2}$.

If we assume that the flip of one spin is evenly distributed as slight misalignments between all n.n spins of the whole system, we get

$$U=-2\Im\sum_{p=1}^{N}\vec{S}_p\cdot\vec{S}_{p+1}=2\,|\,\Im\,|\,NS^2\cos(\pi\pm\tfrac{\pi}{N})=2\,|\,\Im\,|\,NS^2\cos(\pi)\cos(\pi/N)$$
$$=-2\,|\,\Im\,|\,NS^2\cos(\pi/N)=U_0\cos(\pi/N)$$

Thus, for an appreciably large N, we have $U \approx U_0$.

b) Introducing solutions on the form $ue^{i(kpa-\omega t)}$ give

$$-i\omega ue^{i(2pka-\omega t)}=\frac{2i\Im S}{\hbar}\left[2ue^{i(2pka-\omega t)}+ve^{i([2p-1]ka-\omega t)}+ve^{i([2p+1]ka-\omega t)}\right]$$
$$\Rightarrow -\omega u=\frac{4\Im S}{\hbar}\left[u+\frac{v}{2}(e^{-ika}+e^{ika})\right]=\omega_x\left[u+v\cos(ka)\right]$$

for the first differential eqn. Likewise we obtain,

$$\omega v=\omega_x\left[v+u\cos(ka)\right]$$

for the second diff. eqn.

Rearranging

$$\left(\omega_x-\omega\right)u+\left(\omega_x\cos(ka)\right)v=0$$
$$\left(\omega_x\cos(ka)\right)u+\left(\omega_x+\omega\right)v=0$$

Solution for *u* and *v* when the determinant is 0, which gives

$$\left(\omega_x-\omega\right)\left(\omega_x+\omega\right)-\omega_x^2\cos^2(ka)=0$$
$$\Rightarrow \omega^2=\omega_x^2(1-\cos^2(ka))=\omega_x^2\sin^2 ka$$

Thus, the dispersion relation for the anti-ferromagn spin wave is

$$\omega=\omega_x\sin ka$$

c) Domain-formation in a ferromagnet is motivated by reduction of energy if the net magnetisation of the total sample volume can be reduced. Likewise, the costs for forming domains are the spin reversals required to set up either thin or thick domain walls.

In the anti-ferromagnet, there is no net magnetisation in the ground state, and presumably only very weak net magnetisation in thermally agitated states appreciably away from T_N.

Accordingly, formation of domains in antiferromagnets with a perfect lattice would give energy costs to set up domain walls (spin flip), but not give any appreciable benefit in terms of reduction of other terms contributing to the total energy.

Thus, the answer, reasoned from a thorough understanding of the curriculum, would be that domain formation is not energetically favourable in anti-ferromagnets.

Extra-curriculum knowledge: Domains form also in anti-ferromagnets, and the subject is actually a current research topic. The domain formation is believed to be closely associated with various types of lattice defects.