Table – Answers to Problem 4 questions

1	2	3	4	5	6	7	8	9	10
B	A	D	B or D	B	A	C	A	C	D

Problem 1.

a) Equation of motion:

$$m_e \frac{d^2\vec{r}}{dt^2} = -\frac{m_e}{\tau}\frac{d\vec{r}}{dt} - e\vec{E} - e\frac{d\vec{r}}{dt}\times\vec{B}$$

The relaxation time for electron-lattice interactions is even at relatively low temperatures ~ liq. N, 2-3 orders of magnitude shorther than those associated with electron-electron or electron-defect interactions. Thus, resistive loss in metals is in most circumstances properly accounted for taking into account only inelastic collisions between conduction electrons and phonons. The resistivity and phonon exitation typically shows the same temperature dependence.

b) Introducing plane wave solutions into the e.o.m., and neglecting losses, we find for the single electrons

$$-\omega^2 m_e\vec{r} = -e\vec{E} - e\vec{v}_r\times\vec{B}$$

We introduce $\vec{P}$ and transfer the single electron e.o.m. to a mean field variant for the collective electron system

$$-\omega^2 m_e\vec{P} = ne^2(\vec{E} + <\vec{v}_r>\times\vec{B})$$

Whereas the individual electron velocities may be substantial, the net drift velocity $<\vec{v}_r>$ is typically of the order ~ 10^{-5} m/s. Since B/E = v/c, contributions from the magnetic field to the *collective* electron motion may be neglected.

We introduce plane wave solutions and take the long wavelenght limit, i.e. $\vec{k}\to 0$

$$\vec{D} = \varepsilon_0\varepsilon(\omega, k\to 0)\vec{E} = \vec{D} = \varepsilon_0\varepsilon(\omega)\vec{E} = \varepsilon_0\vec{E} + \vec{P}$$

$$\Rightarrow \varepsilon(\omega) = 1 + \frac{\vec{P}_0}{\varepsilon_0\vec{E}_0} \quad \vee \quad \vec{P}_0 = -\frac{ne^2}{\omega^2 m_e}\vec{E}_0$$

$$\Rightarrow \varepsilon(\omega) = 1 - \frac{ne^2}{\varepsilon_0 m_e}\frac{1}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

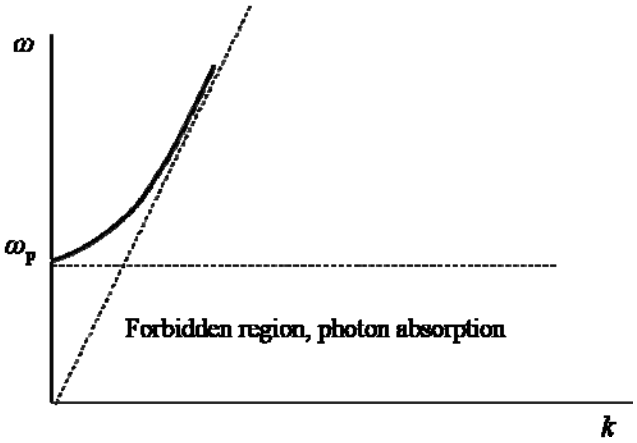
c) From the wave equation we find

$$-\omega^2 D = -\frac{k^2}{\mu_0} E = -c^2 k^2 \varepsilon_0 E \quad \vee \quad D = \varepsilon_0 \varepsilon(\omega,k) E$$
$$\Rightarrow \quad \varepsilon(\omega,k)\omega^2 = c^2 k^2$$

We introduce the plasma response function from b) to arrive at

$$\varepsilon(\omega)\omega^2 = \varepsilon(\infty)(\omega^2 - \omega_p^2) = c^2 k^2$$
$$\Rightarrow \omega = \sqrt{\omega_p^2 - c^2 k^2 / \varepsilon(\infty)}$$

Sketch



i) $\omega < \omega_p$: $\Rightarrow \varepsilon(\omega) < 0; \quad k^2 < 0 \Rightarrow k = i \cdot k'' \Rightarrow E \sim e^{i(\vec{k}\cdot\vec{r}-\omega t)} = e^{-\vec{k}''\cdot\vec{r}} e^{-i\omega t}$, i.e. exponential damping of the wave, equivalent to absorption. Hence, the electromagnetic field does not propagate into the media, and no plasmon exitations takes place (apart from in a shallow region close to the surface).

ii) $\omega = \omega_p$:

$$\Rightarrow \varepsilon(\omega) = 0:$$
$$): \vec{D} = \varepsilon_0 \varepsilon(\omega)\vec{E} = 0 \Rightarrow \vec{E} = -\frac{\vec{P}}{\varepsilon_0}$$

This corresponds to a longitudinal mode, with the plasma polarisation $\vec{P} \,||\, \vec{k}$, where all the free electrons are collectively displaced with respect to the positive ion cores. The associated depolarisation field $\vec{E}$ acts as a restoring force on the gas. The longitudinal mode is only accessible at very low, near-zero *k*-values.

iii) $\omega > \omega_p$: $\Rightarrow \varepsilon(\omega) > 0 \quad \vee \quad \vec{D} \perp \vec{k}$

These are transverse plasma oscillations. $k > 0$, so waves can propagate through the material, i.e. the metal is optically transparent. As the frequency is indreased, the plasma oscillations tend asymptotically toward free-space wave propagation.

Problem 2

The Cooper-pairs can be associated with wave functions on the form $\psi = n^{1/2} e^{i\theta(r)}$, leaving the concentration of pairs $n = \psi^* \psi = \text{constant}$. The generalised momentum operator for a charge q moving in an electromagnetic field is

a) $p_{op} = -i\hbar\nabla - q\vec{A} \Rightarrow v_{op} = \frac{1}{m_{cp}}(-i\hbar\nabla - q_{cp}\vec{A})$

The current density is defined as

$$\vec{j} = nq\vec{v} = q\psi^* v_q \psi = -\frac{2e}{2m_e} n^{1/2} n^{1/2} e^{-i\theta(r)} \left[-i\hbar\nabla + 2e\vec{A}\right] e^{i\theta(r)} = -\frac{en}{m_e}[\hbar\nabla\theta(r) + 2e\vec{A}]$$

where we have taken into account that each cp consists of two electrons.

Taking the curl on both sides, we obtain

$$\nabla \times \vec{j} = -\frac{en}{m_e}[\hbar\nabla \times \nabla\theta(r) + 2e\nabla \times \vec{A}]$$

The first term on r.h.s. is the curl of a gradient and is always 0, thus

$$\nabla \times \vec{j} = -\frac{2e^2 n}{m_e}\nabla \times \vec{A} = -\frac{2e^2 n}{m_e}\vec{B}$$

Apparently, there is a factor 2 too much on the r.h.s., but remember that *n* now is the Cooper pair concentration, whereas the *n* in the London equation is the concentration of *individual* electrons. Accordingly, $n_e = 2n_{cp}$, and our expression for the cooper pair current density is consistent with the London eqn.

b) Recalling that the London equation relates the superconducting current density to the penetration depth of the magnetic field into the superconducting material, which is consistent with the Meissner effect without violating classiscal relations of electromagnetism.

Accordingly, deep into the sc loop $\vec{j} = 0$, leaving $\hbar\nabla\theta(r) = -2e\vec{A}$.

We integrate on both sides over a closed path *C* in the loop interior

$$\oint_C \hbar\nabla\theta \cdot d\vec{l} = \hbar\oint_C \frac{d\theta}{dl} dl = \hbar\oint_C d\theta = -2e\oint_C \vec{A} \cdot d\vec{l} \overset{Stokes}{=} -2e\oint_A \nabla \times \vec{A} \cdot d\vec{A} = -2e\oint_A \vec{B} \cdot d\vec{A}$$

The integral on the l.h.s must correspond to a phase shift $\Delta\theta = s \cdot 2\pi, \quad s = 1, 2, ...$

The last integral on the r.h.s. is recognized as the magnetic flux through the loop, and in result

$$\Phi_m = -\frac{\hbar\pi}{e} \cdot s$$

The negative sign is easily removed, e.g. by redefining the positive direction for $d\vec{A}$

c) The stabilizing energy of the superconducting state corresponds to the critical magnetic field energy that may be surpressed by the material in the superconducting phase. From thermodynamics we find

$$dU = TdS + \mu_0 \vec{M} \cdot d\vec{H}$$

Perfect diamagnetism in type I s.c => $\vec{M} = -\vec{H}$, so $dU = TdS - \mu_0 \vec{H} \cdot d\vec{H}$.

At T = 0 K, we obtain $\Delta U = U(H) - U(0) = \frac{1}{2}\mu_0 H^2$.

When H -> H_C , the critical magnetic field, the normal and superconducting states are in equilibrium and must co-exist with the same internal energy, i.e. $U_N(H_C) = U_{S.C}(H_C)$.

Finally, if the normal state is non-magnetic $\vec{B} = \mu_0(\vec{M} + \vec{H}) \approx \mu_0 \vec{H} \quad \Rightarrow U_N(H) \simeq U_N(0)$

Thus,

$$\Delta U = U_N(0) - U_{S.C}(0) = \tfrac{1}{2}\mu_0 H_C^2 = \frac{B_{\max}^2}{2\mu_0}.$$

The flux corresponding to the critical magentic field is given as $\Phi_{\max} = B_{\max} A_{loop} = \Phi_0 s_{\max}$ when we assume a uniform flux density through the loop.

Finally, we arrive at

$$s_{\max} \le \frac{eA_{loop}\sqrt{2\mu_0 \Delta U}}{\hbar \pi}$$

i. As the temperature is increased, H_C reduces, and accordingly $s_{\max}$ is reduced

ii. The total flux through the loop $\Phi_{tot} = \Phi_{ext} - \Phi_{sc} = \Phi_0 \cdot s$ is restricted in the same fashion as in b), but since the flux from the external field, Φ_{ext} , is not quantizised, Φ_{sc} must adjust itself for Φ_{tot} to meet with the requirements. $s_{\max}$ is not affected.

iii. If the loop is made from a type II superconductor, $s_{\max}$ reaches its maximum at H_{c1}. Any further increase in the magnetic field $H_{c1} < H < H_{c2}$, leads to formation of vortices in the superconducting material, and accordingly migration of flux lines from the loop interior into the metal ring. The flux density in the ring interior remains constant between H_{c1} and H_{c2}.

Problem 3

a) From the Curie Brillouin relation we have

$$M = \frac{N}{V\mu_0} \cdot Jg(JLS)\mu_B\mu_0 \cdot B_J(x) = \frac{N}{V\mu_0}\mu B_J(\frac{\mu H}{k_B T})$$

In the paramagnetic phase, we normally assume weak magnetisation/low fractional alignment. Accordingly, $\mu H << k_B T$, and we can use the small x approximation to arrive at

$$B_J(x) \simeq \frac{2J+1}{2J}\left(\frac{2J}{2J+1}\frac{1}{x} + \frac{1}{3}\frac{2J+1}{2J}x\right) - \frac{1}{2J}\left(\frac{2J}{x} + \frac{1}{3}\frac{1}{2J}x\right)$$
$$= \frac{1}{x} + \frac{(2J+1)^2}{12J^2}x - \frac{1}{x} - \frac{1}{12J^2}x = \frac{4J(J+1)}{12J^2}x = \frac{(J+1)}{3J}x$$

=>

$$M = \frac{N}{V\mu_0}\mu\frac{J+1}{3J}\frac{\mu H}{k_B T} = \frac{N}{V\mu_0}\mu^2\frac{J+1}{3J}\frac{(H_{ext}+\lambda M)}{k_B T}$$
$$= \frac{n\mu_0 g^2\mu_B^2 J(J+1)}{3k_B T}(H_{ext}+\lambda M) = \frac{C}{T}(H_{ext}+\lambda M)$$

): $M = \frac{C}{T - C\lambda}H_{ext}$

We easily verify that M = 0 in the absence of external fields, and that we have paramagnetic alignment when the external field is non-zero. The only issue of concern would be when T -> Cλ, for which the expression becomes singular.

In the ferromagnetic phase, we encounter spontaneous magnetisation and alignment of magnetic moments even in the absence of external fields. It is not anymore reasonable to assume $\mu H << k_B T$, and we must use the full Curie Brillouin relation.

$$M = \frac{N}{V\mu_0}\mu B_J(\frac{\mu H}{k_B T}) = \frac{N}{V\mu_0}\mu B_J(\frac{\mu(H_{ext}+\lambda M)}{k_B T})$$

It is not possible to simplify this any further, or to find analytical solution to the equation. The Brillouin function is however positive for all positive arguments, i.e.

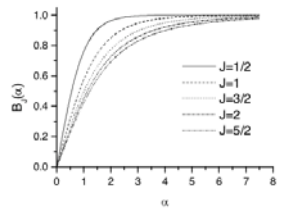


Fig. 7.2 The Brillouin function for the cases J = 1/2 to 5/2

Thus M is non-zero both for $H_{ext}=0$, and $H_{ext} \neq 0$.

b) With the correct answer in a), it is straight forward to find

$$\chi = \frac{\partial M}{\partial H} = \frac{C}{T - \lambda C} = \frac{C}{T - T_C}$$

It is also relatively simple to derive this relationship using the mean field appraoch given in the text, see e.g. Kittel p. 323-324.

The CW law is reasonably precise at higher temperatures, but as one approaches T_C, it starts to deviate from experimental measurements. The reason is that as we approach T_C it may no longer be appropriate to use the small x expansion of the Brillouin function.

c) System in thermal equilibrium at any temperature,

$$\frac{\partial G}{\partial M} = 0 = -H + g_2 M + g_4 M^3$$
$$): H = g_2 M + g_4 M^3$$

For a linear system, the susceptibility

$$\chi = \left[\frac{dH}{dM}\right]_{H=0}^{-1} = (g_2 + 3g_4 M_0^{\,2})^{-1}$$

where M_0 is the equilibrium value of the order parameter in absence of external field.

Thus, for $T > T_C$ where $M_0 = 0$

$$\chi = 1/g_2 = \frac{1}{\gamma(T - T_C)}$$

and for $T < T_C$, system in equilibrium and no external field leaves

$$M(g_2 + M^2 g_4) = 0 => M = 0; \;\; M = \sqrt{-\frac{g_2}{g_4}}$$

Below T_c , $g_2 < 0$, so $g_4 > 0$ gives real non zero solution, and the minimum in

$$G(M) = g_0 + \tfrac{1}{2} g_2 \cdot \left|\frac{g_2}{g_4}\right| + \tfrac{1}{4} g_4 \cdot \left|\frac{g_2}{g_4}\right|^2 = g_0 - \tfrac{1}{2} \cdot \frac{g_2^2}{g_4} + \tfrac{1}{4} \cdot \frac{g_2^2}{g_4} = g_0 - \tfrac{1}{4} \cdot \frac{g_2^2}{g_4} < g_0$$

Thus, below T_c : $M_0 = \pm M = \pm\sqrt{-\frac{g_2}{g_4}}$

$$\chi = \frac{1}{g_2 + 3g_4(\frac{-g_2}{g_4})} = \frac{1}{-2g_2} = \frac{1}{2\gamma(T_C - T)}$$

Direct comparison with the paramagnetic suceptibility in b), reveals $\gamma = 1/C$