Problem 1.

a) Equation of motion:

$$M\frac{d^2\Delta\vec{u}}{dt^2} = q\vec{E}_{ext} - F_{TO} = q\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} - \omega_{TO}^2 M\Delta\vec{u}$$

No additional polarisation of the ion cores by the field, such that the field experienced by the ions is equivalent to the external field, i.e. no local field contribution ($\vec{E}^* = \vec{E}_{ext} + \vec{E}_{loc}$)

In the long wave length limit (k-> 0), we assume no spatial variation of the field, i.e. the instantaneous ion-pair displacements are the same anywhere inside the sample.

Thus, the total polarisation caused by such ion displacements (dipole moment per unit volume) is:

$$\vec{P} = nq\Delta u$$

Hence, we may express a collective e.o.m. in terms of field quantities:

$$\frac{d^2\vec{P}}{dt^2} = \frac{nq^2}{M}\vec{E}_{ext} - \omega_{TO}^2\vec{P}$$

The time dependence of the polarisation is determined entirely by the time structure of the external electric field. Accordingly,

$$(\omega_{TO}^2 - \omega^2)\vec{P} = \frac{nq^2}{M}\vec{E}_{ext}$$

The long-λ limit is generally a very reasonable approach to the phonon-photon coupling. This is easily seen by comparing the electromagnetic dispersion relation ω=ck to typical phonon frequencies, implying that such resonance occur predominantly at very small values of k.

b) Wave eqn.

$$\nabla^2\vec{E} = \mu_0\frac{\partial^2\vec{D}}{\partial t^2} \qquad \Rightarrow -k^2\vec{E} = -\mu_0\omega^2\vec{D} = -\frac{\omega^2}{c^2}(\vec{E} - \frac{\vec{P}}{\varepsilon_0})$$

$$\Rightarrow \frac{\omega^2}{\varepsilon_0}\vec{P} = (\omega^2 - c^2k^2)\vec{E}$$

Dispersion relation (general):

$$\begin{vmatrix} (\omega_{TO}^2 - \omega^2) & -\frac{nq^2}{M} \\ \omega^2/\varepsilon_0 & (c^2k^2 - \omega^2) \end{vmatrix} = \omega^4 - \omega^2(c^2k^2 + \omega_{TO}^2 + \frac{nq^2}{M\varepsilon_0}) + c^2k^2\omega_{TO}^2 = 0$$

In the short wavelength limit, we get:

$$\omega^4 - \omega^2(\omega_{TO}^2 + \frac{nq^2}{M\varepsilon_0}) = \omega^2(\omega^2 - \omega_{TO}^2 + \frac{nq^2}{M\varepsilon_0}) = 0$$

$$\Rightarrow \omega = 0 \quad \wedge \quad \omega = \pm\sqrt{\omega_{TO}^2 + \frac{nq^2}{M\varepsilon_0}}$$

Since ω>0 for any wave, the positive root is the only possible non-trivial solution.

c) The long-λ dielectric function

$$\varepsilon_0\varepsilon\vec{E} = \varepsilon_0\vec{E} + \vec{P} \Rightarrow \varepsilon(\omega) = 1 + \frac{\vec{P}}{\varepsilon_0\vec{E}} = 1 + \frac{nq^2}{M\varepsilon_0}\frac{1}{\omega_{TO}^2 - \omega^2}$$

The optical dielectric constant ε(∞) ≈1, and the static dielectric constant

$$\varepsilon(0) = 1 + \frac{nq^2}{M\varepsilon_0}\frac{1}{\omega_{TO}^2} = \varepsilon(\infty) + \frac{nq^2}{M\varepsilon_0}\frac{1}{\omega_{TO}^2}$$

Thus, we may express the dielectric function

$$\varepsilon(\omega) = \varepsilon(\infty) + [\varepsilon(0) - \varepsilon(\infty)]\frac{\omega_{TO}^2}{\omega_{TO}^2 - \omega^2} = \frac{\omega_{TO}^2\varepsilon(0) - \omega^2\varepsilon(\infty)}{\omega_{TO}^2 - \omega^2}$$

The special situation where ε -> 0 correspond to a situation where the incident photon frequency approaches that associated with a longitudinal optical phonon (even if coupling to longitudinal phonons is impossible due to the transversal nature of electromagnetic fields), i.e.

$$\varepsilon(\omega \to \omega_{LO}) = \omega_{TO}^2\varepsilon(0) - \omega_{LO}^2\varepsilon(\infty) = 0$$

$$\Rightarrow \frac{\varepsilon(0)}{\varepsilon(\infty)} = \frac{\omega_{LO}^2}{\omega_{TO}^2}$$

Problem 2.

a) Introduce the $\psi_j = n_j^{1/2}e^{i\theta_j}$ solutions into the time depedent Schr.eqns. When tunnelling is assumed to take place, both amplitudes and phases will be time dependent quantities

$$\frac{i\hbar}{2}n_1^{-1/2}\frac{\partial n_1}{\partial t}e^{i\theta_1} - \hbar\frac{\partial\theta_1}{\partial t}n_1^{1/2}e^{i\theta_1} = En_1^{1/2}e^{i\theta_1} + \hbar Tn_2^{1/2}e^{i\theta_2} \qquad (2)$$

$$\frac{i\hbar}{2}n_2^{-1/2}\frac{\partial n_2}{\partial t}e^{i\theta_2} - \hbar\frac{\partial\theta_2}{\partial t}n_2^{1/2}e^{i\theta_2} = En_2^{1/2}e^{i\theta_2} + \hbar Tn_1^{1/2}e^{i\theta_1} \qquad (3)$$

Multiply the eqns with their respective complex conjugated wave functions, $\psi_j^* = n_j^{1/2}e^{-i\theta_j}$, and rearrange

$$\frac{1}{2}\frac{\partial n_1}{\partial t} + in_1\frac{\partial \theta_1}{\partial t} = -iTn_1^{1/2}n_2^{1/2}e^{i(\theta_2-\theta_1)} - \frac{i}{\hbar}En_1 \qquad (2)$$

$$\frac{1}{2}\frac{\partial n_2}{\partial t} + in_2\frac{\partial \theta_2}{\partial t} = -iTn_1^{1/2}n_2^{1/2}e^{-i(\theta_2-\theta_1)} - \frac{i}{\hbar}En_2 \qquad (3)$$

Separate real and imaginary terms

$$\frac{\partial n_1}{\partial t} = 2Tn_1^{1/2}n_2^{1/2}\sin(\theta_2-\theta_1) = -\frac{\partial n_2}{\partial t}$$

$$i\frac{\partial \theta_1}{\partial t} = -iTn_1^{-1/2}n_2^{1/2}\cos(\theta_2-\theta_1) - \frac{i}{\hbar}E$$

$$i\frac{\partial \theta_2}{\partial t} = -iTn_1^{1/2}n_2^{-1/2}\cos(\theta_2-\theta_1) - \frac{i}{\hbar}E$$

With the two superconductors identical $n_1 = n_2$

$$\frac{\partial n_1}{\partial t} = 2Tn\sin(\theta_2-\theta_1) = -\frac{\partial n_2}{\partial t}$$

$$\frac{\partial \theta_1}{\partial t} = -T\cos(\theta_2-\theta_1) - \frac{E}{\hbar} = \frac{\partial \theta_2}{\partial t}$$

The second eqn. explicitly gives

$$\frac{\partial \theta_2}{\partial t} - \frac{\partial \theta_1}{\partial t} = \frac{\partial(\theta_2-\theta_1)}{\partial t} = 0 \Rightarrow \quad \theta_2-\theta_1 = \text{const}$$

The current flow from superconductor 1 to 2 is proportional to $\frac{\partial n_2}{\partial t} = -\frac{\partial n_1}{\partial t}$, (dim: # CPs/(m^3s), i.e. equivalent to the current density divided by unit length and the charge of the cooper pair) so the current density of cooper pairs through the barrier could be expressed on the form

$$\vec{j} = \vec{j}_0\sin(\theta_2-\theta_1)$$

From the relation it is seen that the Josephson DC-current is driven by the phase difference between the two Cooper-pair wave functions (or in other words the difference in cooper pair concentrations on each side of the barrier). If the two are in phase, there is no cooper pair current through the barrier.

b) The two junctions make up a closed superconducting loop, and accordingly the magnetic flux that passes through the loop will be quantized. The flux can be expressed

$$\varphi = \varphi_0 \cdot s = \frac{\hbar\pi}{e}\cdot s$$

The expression for flux quantisation through the superconducting loop came as a result of the Meissner effect, and must therefore be contained in the London equation. We do not need a full deduction here. It is sufficient to realize that for a full loop the phase shift, δ, of the current must be a multiple of 2π, i.e

$$\delta = 2\pi \cdot s = \frac{2e}{\hbar}\varphi$$

Two identical parallel junctions on the right and left hand side of the loop, with phase shifts δ_r and δ_l, respectively. If the direction indicated in the figure is taken as positive, we have for a full (counter clockwise) loop

$$\delta = \frac{2e}{\hbar}\varphi = \delta_r - \delta_l$$

The last equation can be rephrased to yield

$$\delta_r = \delta_0 + \frac{e}{\hbar}\varphi \qquad \delta_l = \delta_0 - \frac{e}{\hbar}\varphi$$

Thus, the total current density along the indicated direction, assuming two identical Josephson DC-junctions in parallel is

$$j_{tot} = j_r + j_l = j_0 \sin(\delta_0 - \frac{e}{\hbar}\varphi) + j_0 \sin(\delta_0 + \frac{e}{\hbar}\varphi) = 2 j_0 \sin\delta_0 \cos(\frac{e}{\hbar}\varphi)$$

The last expression is analogous to that found for two-source interference in optics, see e.g. any text book in basic univ. physics. The device is a so-called Superconducting Quantum Interference Device or SQUID. It is used mainly to carry out very precise measurements of magnetic fields, alternatively to measure very weak magnetic fields.

It does not matter for the basic functionality of the device if the superconductors are of type I or type II. In the latter case flux quantisation within the loop still apply until H_{C2} is reached. Yet, if the sc's are of type II, there will be a small difference. In the case of a type I, the device may work to measure flux increase for fields monotonically up to H_C. For a type II, the maximum flux within the loop will be limted by H_{C1}, whereas for any further increase of the field between H_{C1} and H_{C2} the loop still functions but the flux lines corresponding to increase of the field, will migrate into the sc's to form vortices.

Problem 3

a) From the two ions we get
<u>Mn^{2+}</u>: $3d^5$. Crystal field splitting => $J = S = 5/2$, $L = 0$.

$$g(JLS) = \frac{3}{2} + \frac{1}{2}\frac{S(S+1) - L(L+1)}{J(J+1)} = 2$$

$$p = g(JLS)(J(J+1))^{1/2} = 2(\frac{5}{2}(\frac{7}{2}))^{1/2} \approx 5.916$$

<u>Co^{3+}</u>: $3d^6$. Crystal field splitting => $J = S = 2$, $L = 0$.

$$g(JLS) = \frac{3}{2} + \frac{1}{2}\frac{S(S+1) - L(L+1)}{J(J+1)} = 2$$

$$p = g(JLS)(J(J+1))^{1/2} = 2(2\cdot 3)^{1/2} \approx 4.900$$

For the total system (along c), the moments of Co^{3+} cancel, and we are left with a net contribution from Mn^{2+}, i.e. $p_{AB} = p_{Mn^{2+}} \approx 5.916$.

b) No external field. The sign of the exchange coeffs determine if parallel or antiparallel alignment is favourable, and generally negative sign favours anti-parallel moments (logic, since the effective field should be opposite to the neighbour site magnetisation). Since all exchange should be antiparallel (given in the text), we are stuck with

Exchange field site A:
$$\vec{H}_A = -\lambda_{AA}\vec{M}_A - \lambda_{AB}\vec{M}_B$$

Exchange field site B:
$$\vec{H}_B = -\lambda_{BB}\vec{M}_B - \lambda_{AB}\vec{M}_A$$

Energy densities:
$$dU = -\mu_0 \vec{M} d\vec{H} = -\mu_0 \vec{M}_i d\vec{M}_j \frac{d\vec{H}_i}{d\vec{M}_j}$$

$$U = -\frac{1}{2}\mu_0 \vec{M}_i \vec{M}_j \frac{d\vec{H}_i}{d\vec{M}_j} = -\frac{1}{2}\mu_0 (M_A M_A \frac{dH_A}{dM_A} + \vec{M}_A \cdot \vec{M}_B (\frac{dH_A}{dM_B} + \frac{dH_B}{dM_A}) + M_B M_B \frac{dH_B}{dM_B})$$

=>

$$U = -\frac{1}{2}\mu_0(-\lambda_{AA}M_A^2 - 2\lambda_{AB}\vec{M}_A \cdot \vec{M}_B - \lambda_{BB}M_B^2) = \frac{1}{2}\mu_0(\lambda_{AA}M_A^2 + 2\lambda_{AB}\vec{M}_A \cdot \vec{M}_B + \lambda_{BB}M_B^2)$$

): Energy minimum with antiparallel magnetisation in A and B sites.

c) Only AB interactions. Effective fields

A-sites: $\vec{H}_A = \vec{H}_{ext} - \lambda_{AB}\vec{M}_B$
B-sites: $\vec{H}_B = \vec{H}_{ext} - \lambda_{AB}\vec{M}_A$

Paramagnetic response with weak magnetisation of both A and B sites, so Curies law may be applied to describe the mean field temperature dependent paramagnetic responses of the A and B sites

$$M_A T = C_A H_A = C_A (H_{ext} - \lambda_{AB} M_B)$$
$$M_B T = C_B H_B = C_B (H_{ext} - \lambda_{AB} M_A)$$

In zero external field (H_{ext}=0) these equations have non-zero solutions for M_A and M_B when

$$\begin{vmatrix} T & \lambda_{AB}C_A \\ \lambda_{AB}C_B & T \end{vmatrix} = T^2 - \lambda_{AB}^2 C_A C_B = 0$$

Since this is inconsistent with a paramagnetic response, we conclude that it corresponds to a temperature, T_C, where the moments order spontaneously into a ferromagnetic phase, i.e

$$T_C = \lambda_{AB}\sqrt{C_A C_B}$$

In the paramagnetic phase the system must obey a Curie-Weiss type of relationship. Assume weak magnetisation, so that a linear susceptibility holds (except very near T_C):

$$\chi = \frac{\partial(M_A + M_B)}{\partial H_{eff}} \approx \frac{\partial(M_A + M_B)}{\partial H_{ext}} = \frac{C_A}{T}(1-\lambda\frac{\partial M_B}{\partial H_{ext}}) + \frac{C_B}{T}(1-\lambda\frac{\partial M_A}{\partial H_{ext}})$$

$$= \frac{C_A}{T}(1-\lambda\frac{C_B}{T}(1-\lambda\frac{\partial M_A}{\partial H_{ext}})) + \frac{C_B}{T}(1-\lambda\frac{C_A}{T}(1-\lambda\frac{\partial M_B}{\partial H_{ext}}))$$

$$= \frac{1}{T^2}(C_A T - \lambda C_A C_B + T_c^2\frac{\partial M_A}{\partial H_{ext}}) + \frac{1}{T^2}(C_B T - \lambda C_A C_B + T_c^2\frac{\partial M_B}{\partial H_{ext}})$$

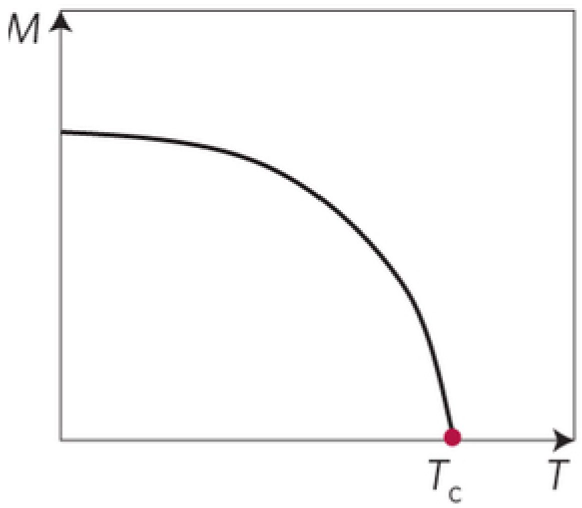
$$=> \frac{\partial(M_A + M_B)}{\partial H_{ext}}\left(1-\frac{T_c^2}{T^2}\right) = \frac{(C_A + C_B)T - 2\lambda C_B C_A}{T^2}$$

$$=> \chi = \frac{(C_A + C_B)T - 2\lambda C_B C_A}{T^2 - T_c^2}$$

The result looks messy, but yields a paramagnetic susceptibility similar to that of a ferromagnet.

The ferrimagnet has an antiferromagnetic arrangement of unbalanced magnetic moments, and will therefore yield a temperature dependent magentisation similar to that of a ferromagnet in zero external field (i.e. in the perfect ferrimagn. lattice at T=0 there is a net contribution per unit AB of 5.91 Bohr magnetons, and these are parallel to one another.)

In <u>zero external field</u> we should therefore expect an archetype 2nd order response function



Since the array itself represents and antiferromagnetic ordering, the ferrimagnetic response function may differ substantially from the ferromagnetic one in external fields. Furthermore, the spin waves of the ferrimagnetic lattice are of an antiferromagnetic nature (but cases with unbalanced moments extends well beyond our curriculum).

<u>Last question:</u>
Circumstances where M(T) ->0 (well off $T_{C,}$ of course). According to the simple models covered within the curriculum, this should not be possible. However, one could imagine a hypothetical case for a second order transition with relatively weak opposing moments close to T_C. If the opposing moments evolve differently with temperature, there could be a point below T_C where they cancel out completely like in a perfect antiferromagnet. Systems like this do exist.