Solutions Exam 2016

Problem 1.

a) E.o.m for an individual electron:

$$m_e \frac{d^2\vec{r}}{dt^2} = -\frac{m_e}{\tau}\frac{d\vec{r}}{dt} - e\vec{E} - e\frac{d\vec{r}}{dt}\times\vec{B}$$

Introduce plane waves to obtain

$$-\left(i\omega\frac{m_e}{\tau} + \omega^2 m_e\right)\vec{r}_0 = -e\vec{E}_0 - e\vec{v}_r \times \vec{B}_0 = -e\vec{E}_0 - \frac{e\hbar}{m_e}\vec{k}\times\vec{B}_0$$

where we have introduced the free electron velocity expressed in terms of $\vec{k}$

For the whole free electron plasma, we introduce the collective response function $\vec{P} = -ne\vec{r}$, and furthermore restrict ourselves to the long wavelength limit, i.e. $\vec{k} \to 0$, so that the e.o.m. transfers to the following mean field e.o.m for the whole plasma:

$$\left(\omega^2 m_e + i\omega\frac{m_e}{\tau}\right)\vec{P}_0 = -ne^2\vec{E}_0 - \frac{ne^2\hbar}{m_e}\vec{k}\times\vec{B}_0 \approx -ne^2\vec{E}_0$$

Employing Maxwell equations, we find

$$\vec{D} = \varepsilon_0\varepsilon(\omega,k\to 0)\varepsilon_r\vec{E} = \vec{D} = \varepsilon_0\varepsilon(\omega)\varepsilon_r\vec{E} = \varepsilon_0\vec{E} + \vec{P}$$

$$\Rightarrow \varepsilon(\omega) = 1 + \frac{\vec{P}_0}{\varepsilon_0\varepsilon_r\vec{E}_0} \quad \vee \quad \vec{P}_0 = -\frac{ne^2}{\frac{\omega m_e}{\tau}(\omega\tau + i)}\vec{E}_0$$

$$\Rightarrow \varepsilon(\omega) = 1 - \frac{ne^2}{\varepsilon_0\varepsilon_r m_e}\frac{\tau}{\omega(\omega\tau + i)} = 1 - \frac{ne^2}{\varepsilon_0\varepsilon_r m_e}\frac{\tau^2}{\omega^2\tau^2 + \omega\tau i} = 1 - \frac{\omega_p^2\tau^2}{\omega^2\tau^2 + \omega\tau i}$$

b) Employing the wave equation, we find

$$-\omega^2 D = -\frac{k^2}{\mu_0}E = -c^2k^2\varepsilon_0 E \quad \vee \quad D = \varepsilon_0\varepsilon(\omega,k)\varepsilon_r E$$

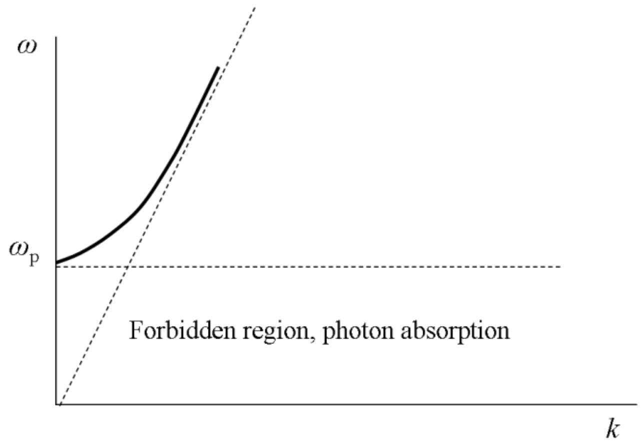
$$\Rightarrow \quad \varepsilon(\omega,k)\omega^2 = \frac{c^2k^2}{\varepsilon_r}$$

Introduce the plasma response function from a) without the imaginary loss term, to arrive at

$$\varepsilon(\omega)\omega^2 = \omega^2 - \omega_p^2 = \frac{c^2k^2}{\varepsilon_r}$$

$$\Rightarrow \omega = \sqrt{\omega_p^2 - c^2k^2/\varepsilon_r}$$

Sketch



For $\omega = \omega_p$:

$$\Rightarrow \varepsilon(\omega) = 0:$$

$$): \vec{D} = \varepsilon_0 \varepsilon(\omega)\vec{E} = 0 \Rightarrow \vec{E} = -\frac{\vec{P}}{\varepsilon_0}$$

This corresponds to a longitudinal oscillation mode, with all the free electrons collectively displaced with respect to the positive ion cores, such that the associated depolarisation field $\vec{E}$ acts as a restoring force on the gas.

Since the longitudinal mode implies plasma polarisation $\vec{P} \parallel \vec{k}$, it cannot be excited by a (transversal) electromagnetic wave. It may, however, be excited by charged particles, e.g. electrons.

c) Express the fields on each side of the interface via the potential gradient:

$$\vec{D}_i = \varepsilon_i \vec{E}_i = -\varepsilon_i \nabla \phi_i, \quad i = 1,2$$

By the boundary conditions for the displacement field, we get

$$D_{1z} = -\varepsilon_1 \frac{\partial}{\partial z} \phi_1(x,z)\Big|_{z=0} = \varepsilon_1 kA\cos(kx)$$

$$D_{2z} = -\varepsilon_2 \frac{\partial}{\partial z} \phi_2(x,z)\Big|_{z=0} = -\varepsilon_2 kA\cos(kx)$$

In $z = 0$:

$$D_{1z} = D_{2z} \Rightarrow \varepsilon_1 = -\varepsilon_2$$

$$\Rightarrow \left(1 - \frac{\omega_{p1}^2}{\omega^2}\right) = \left(-1 + \frac{\omega_{p2}^2}{\omega^2}\right)$$

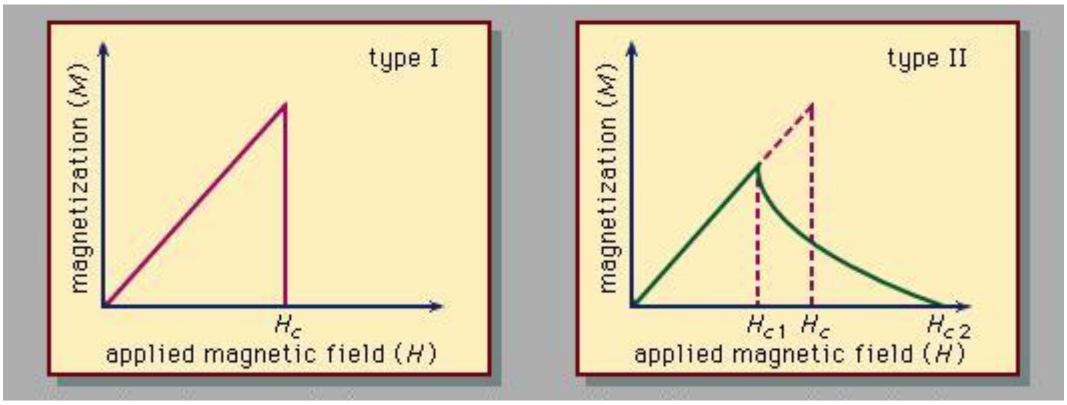
$$\Rightarrow 2\omega^2 = \omega_{p1}^2 + \omega_{p2}^2$$

$$\Rightarrow \omega = \left[\tfrac{1}{2}\left(\omega_{p1}^2 + \omega_{p2}^2\right)\right]^{1/2}$$

Concerning the free electron model versus a more realistic nearly free electron model for the plasma behaviour, it suffices to compare the free electron band structure to the one we obtain by employing the Bloch wave formalism. The discrepancies between the two approaches are found in the vicinity of the Brillouin zone edges, where band gaps open up when the periodic potential is accounted for. In other words, the differences between the two approaches are located mainly at relatively large *k*-values. Since the long wave length model is limited to analysis for *k* values close to the Brillouin zone center, there is no discernible difference between a free electron approach and a more sophisticated model for the conduction electrons.

Problem 2

a) Sketch



For a type I s.c., magnetic flux is expelled (Meissner effect), causing a perfect diamagnetic response, i.e. M=-H, until a critical field H_C is reached. At H_C the supeconducting state becomes unstable, and the conductor returns to normal state.

For a type II s.c. the behaviour is identical to type 1 up to $H=H_{c1.}$ At this critical field the s.c. enters into the so-called vortex state. The bulk material remains superconducting, but allow for small islands (or vortices) to penetrate into the material. The vortex interior is in normal state and thus allow for magnetic flux to penetrate the material. The external field can be increased up to H_{c2} by creating more and more vortices. At $H_{c2,}$ however, the density of vortices is saturated. Forming one more vortex would imply a density which exceeds the absolute minimum average vortex separation distance, i.e. the coherence length. The latter is the minimum spatial distance required for the existence of Cooper pairs. Thus at H_{c2} bulk superconductivity breaks down for the type II s.c., and it returns to normal state.

b) From Maxwell, $\nabla\times\vec{B}=\mu_0(\vec{j}+\frac{\partial\vec{D}}{\partial t}+\nabla\times\vec{M})=\mu_0\vec{j}$, assuming no displacement currents or Amperian currents/magnetisation currents.

Take the curl once more, and introduce the superconducting current density

$$\nabla\times\nabla\times\vec{B}=\nabla\cdot(\nabla\cdot\vec{B})-\nabla^2\vec{B}=-\nabla^2\vec{B}=\mu_0(\nabla\times\vec{j})$$

$$=-\mu_0\frac{e}{m_e}\,n_{c.p.}(\hbar\nabla\times\nabla\theta(r)+\ 2e\nabla\times\vec{A})=-\mu_0\frac{2e^2n_{c.p.}}{m_e}\vec{B}$$

$$\Rightarrow\nabla^2\vec{B}=\frac{2e^2\mu_0 n_{c.p.}}{m_e}\vec{B}=\lambda_L^{-2}\vec{B}$$

We have a straight, cylindrical type I superconductor. The London equation applies to both the B-field and the superconducting current density, as it relates the two in accordance with both the Meissner effect and Amperes law.

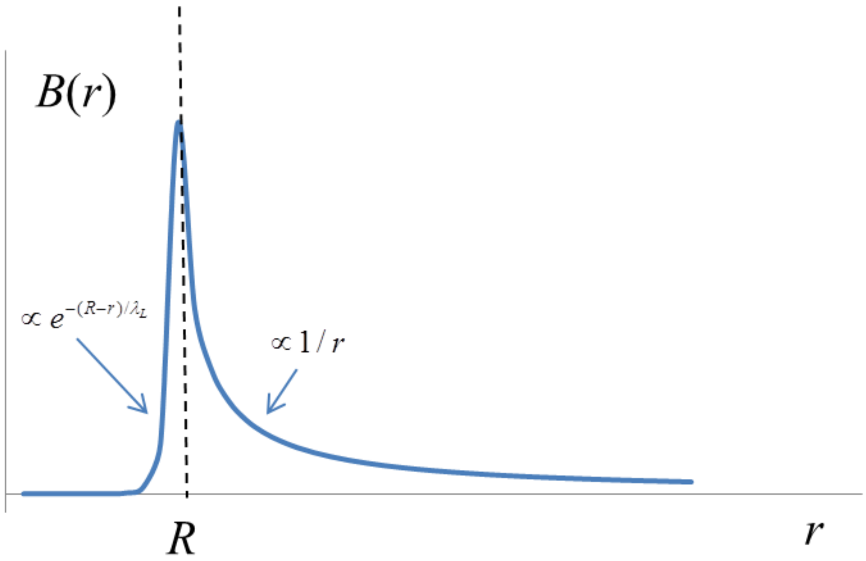
Accordingly, the superconducting current density must also satisfy $\nabla^2\vec{j}=\lambda_L^{-2}\vec{j}$. A solution to these equations inside the cylindrical cross section superconductor is

$$B(r)=B(R)e^{-(R-r)/\lambda_L}\quad\text{and}\quad j(r)=j(R)e^{-(R-r)/\lambda_L}$$

On the outside of the conductor (r > R),

$$\oint_c\vec{B}\cdot d\vec{l}=\mu_0 I_{s.c.}\Rightarrow B(r)=\frac{\mu_0 I_{s.c.}}{2\pi r}$$

Sketch:



λ_L is the London penetration depth, i.e. a parameter that describes partial penetration of the B-field into the superconductor, and also the spatial region in which a superconducting current may flow if consistency with the Meissner effect and Maxwells eqns is to remain valid.

c) At equilibrium g should attain its minimum value wrt any of its variables, incl. |ψ|

$$\frac{\partial g(T,|\psi|)}{\partial|\psi|}=2|\psi|(\gamma(T-T_C)+2g_4|\psi|^2)=0$$
$$\Rightarrow|\psi|=0 \quad \text{or} \quad \mathrm{n}_{c.p}(T)=|\psi|^2=\frac{\gamma(T_C-T)}{2g_4}$$

Type II eqn:

$$g(\psi(\mathrm{r}))\ \text{-}\ g_0 \propto\ \text{-}\frac{|\psi|^2(r)}{|\psi|_{eq}^{\ 2}}+\frac{|\psi|^4(r)}{2|\psi|_{eq}^{\ 4}}\ +\ c_{GL}^2\frac{|\nabla\psi|^2(r)}{|\psi|_{eq}^{\ 2}}$$

While the two first terms on the right hand side are dimensionless, the last term $\frac{|\nabla\psi|^2}{|\psi|^2}$ has dimensions 1/m^2. Accordingly c_{Gl} must have dimension m, and be a length.

Consider equilibrium solution for the transition region

$$\frac{\partial\Delta g}{\partial|\psi|(r)}\ =0=2\cdot\frac{|\psi|(r)}{|\psi|_{eq}^4}\left(|\psi|^2(r)-|\psi|_{eq}^2\right)+\frac{c_{GL}^2}{|\psi|_{eq}^2}\frac{\partial}{\partial|\psi|(r)}\nabla^2(|\psi|(r)\cdot|\psi|(r))$$
$$\Rightarrow c_{GL}^2\nabla^2|\psi|(r)=-\frac{|\psi|(r)}{|\psi|_{eq}^2}\left(|\psi|^2(r)-|\psi|_{eq}^2\right)$$
$$\Rightarrow c_{GL}^2\nabla^2 n^{1/2}(r)=-\frac{n^{1/2}(r)}{n_{eq}}\left(n(r)-n_{eq}\right)$$

Interpreting the equation, we see that it expresses the difference in free energy density in a transition region from an n.s. region with energy g_0 into a bulk s.c. region with g(T, |ψ|) = constant and $|\psi|=|\psi|_{eq}$, just like around vortex lines in the type ii conductor.
The characteristic length c_{Gl} must be the so-called coherence length.

Temperature dependence:

For the left hand side to balance the temperature dependence of the right side,

$$c_{GL}^2\propto\frac{1}{n_{eq}}\Rightarrow c_{GL}\propto(T_c-T)^{-1/2}$$

Problem 3

a) $3d^3$ => S=3/2, L=2+1+0=3, J=|L-S|= 3/2, $g(JLS)=\frac{3}{2}+\frac{1}{2}\frac{\frac{3}{2}\cdot\frac{5}{2}-3\cdot4}{\frac{3}{2}\cdot\frac{5}{2}}=0.4$

$\Rightarrow p = g(JLS)[J(J+1)]^{1/2} = 0.77$

Measurements indicate a p value ~ 5 times larger. Since we are dealing with moments from the 3d shell, it may be that the crystalline form is associated with so-called quenching of the orbital angular momentum. In that case:

S=3/2, L=0, J=|S= 3/2, $g(JLS) = \frac{3}{2} + \frac{1}{2}\frac{\frac{3}{2}\cdot\frac{5}{2} - 3\cdot 4}{\frac{3}{2}\cdot\frac{5}{2}} = 0.4$

$\Rightarrow p = 3.87$, and 3.87/0.77 = 5.02, OK.

b) The effective microscopic field may be expressed

$$\vec{H}_{eff} = \vec{H}_{ext} - \frac{1}{g\mu_B\mu_0}\sum_{j\neq i}\Im(\Delta\vec{r}_{ij})\cdot\vec{S}_j$$

With all N $\vec{S}_j$ spins assumed equivalent and replaced by the thermal averaged value, $<\vec{S}_j>_T$, the mean field magnetisation of the system is

$$\vec{M} = \frac{1}{V}\sum_j <\vec{\mu}_j>_T = \frac{-g\mu_B}{V}\sum_j <\vec{S}_j>_T = -\frac{N}{V}g\mu_B <\vec{S}_j>_T$$

Thus, the mean field approximation to the effective field can be expressed as

$$\vec{H}_{eff} = \vec{H}_{ext} + \lambda\vec{M}; \qquad \lambda = \frac{\sum_{i\neq j}\Im(<r_{ij}>_T)}{n\mu_0\mu_B^2 g^2} = \frac{2<\Im>_T}{n\mu_0\mu_B^2 g^2}$$

which corresponds to the Weiss molecular field.

For the paramagnetic response above T_C, we assume weak magnetisation, and may therefore use the Curie-Brillouin relationship

$$M = \frac{N}{V}\cdot g(JLS)J\mu_B\cdot B_J\left(\frac{g(JLS)J\mu_B\mu_0 H}{k_B T}\right) = \frac{n}{\mu_0}\mu B_J\left(\frac{\mu H}{k_B T}\right)$$

in the small argument limit $\mu H << k_B T$ for the Brillouin function, i.e.

$$\coth(x) = \frac{1}{x} + \frac{1}{3}x \ \Rightarrow B_J(x) \simeq \frac{(J+1)}{3J}\cdot x$$

We substitute H with the Weiss field, and use the Curie-law to arrive at

$$M = \frac{n\mu^2}{\mu_0}\frac{(J+1)}{3Jk_B T}\left(H_{ext} + \lambda M\right) = \frac{n\mu_0 g^2\mu_B^2 S(S+1)}{3k_B T}\left(H_{ext} + \lambda M\right) = \frac{C}{T}\left(H_{ext} + \lambda M\right)$$

Thus,

$$M(T - C\lambda) = CH_{ext}$$

$$=> \chi = \frac{\partial M}{\partial H_{ext}} = \frac{C}{T - T_C}$$

with $T_C = C\lambda = \frac{n\mu_0 g^2 S(S+1)\mu_B^2}{3k_B} \frac{2 < \Im >_{T_C}}{n\mu_0 \mu_B^2 g^2} = \frac{2S(S+1) < \Im >_{T_C}}{3k_B}$

c) $M = \frac{n}{\mu_0} \mu B_J (\frac{\mu H_{eff}}{k_B T})$

Enter mean field and microscopic versions of H_{eff}, with $H_{ext} = 0$

Mean field: $M = \frac{n}{\mu_0} \mu B_J (\frac{\mu \lambda M}{k_B T}) = \frac{n}{\mu_0} \mu B_J (\frac{2 < \Im >_T S}{n \mu_B g k_B T} \cdot M)$

Microscopic: $M = \frac{n}{\mu_0} \mu B_J (\frac{\vec{\mu} \cdot \vec{H}_{eff}}{k_B T}) = \frac{n}{\mu_0} \mu B_J \left(\frac{-\sum_{i \neq j} \Im(\Delta \vec{r}_{ij}) \vec{S}_i \cdot \vec{S}_j}{k_B T} \right)$

Difference: Microscopic version retains the full energy accountancy of spin-wave/Magnon excitations, whereas the mean field version only characterise the net behaviour. The microscopic version also accounts for possible fluctuations in the spin wave (e.g. spin directions and interatomic distances/exchange intergrals), whereas the mean field version only relates to thermal equilibrium values….