TFY4245 Adv. solid state physics, Solutions Exam 26/5 2018

Problem 1 - Plasmons

a) Long wavelength limit: $k->0 => \epsilon(\vec{k},\omega)->\epsilon(\omega)$. Let the x-axis be parallel to the applied electric field, $\vec{E}=\vec{E_0}e^{i\omega t}$. The displacement of a single electron caused by the field, with respect to a neutral (equilibrium) position in $x=0$ is given by:

$$m\frac{d^2x}{dt^2}=-eE_0e^{i\omega t}$$

.

Thus, solutions should be on the form $x=x_0e^{i\omega t}$, and from substitution into the e.o.m. above, one gets:

$$x_0=\frac{eE_0}{m\omega^2}$$

The electron gas as a whole has a dipole moment per unit volume, or a polarisation, which by definition is:

$$P=-nex=-\frac{ne^2}{m\omega^2}E$$

The (relative) dielectric response is then

$$\epsilon=\frac{D}{\epsilon_0 E}=\frac{\epsilon_0 E+P}{\epsilon_0 E}=1-\frac{ne^2}{\epsilon_0 m}\frac{1}{\omega^2}=1-\frac{\omega_p^2}{\omega^2}$$

The ion core contribution illustrated in the figure, shows a relatively high value of ϵ_{ic} at low frequencies, falling off as ω increases, asymptotically approaching a constant value labled $\epsilon_{ic}(\infty)$ in the figure and text. The latter reflects a so-called high-frequency dielectric constant. At low frequencies, ($\sim$ IR), the dielectric function reflects a coupling between the electric field and phonons (thermal vibrations of the ion cores at ambient temperatures are in the IR-domain). As the frequency of the field is increased, we gradually loose the E-field-phonon coupling, since the field ocsillations become too rapid to resonate with the phonons. The asymptotic value $\epsilon_{ic}(\infty)$ reflects a correction to the vacuum permittivity due to the fact that the core electrons (weakly) adapt to the incoming field oscillations.

For the frequency range relevant for plasmon exitations ($\omega>\omega_p$), the total relative dielectric function for the plasma becomes:

$$\epsilon_{plasma}(\omega)=\epsilon_{ic}(\infty)-\frac{\omega_P^2}{\omega^2}=\epsilon_{ic}(\infty)(1-\frac{\bar{\omega}_p^2}{\omega^2}),$$

where $\bar{\omega}_p^2=\frac{\omega_p^2}{\epsilon_{ic}(\infty)}$

b) From the wave equation, with an electric field on the form $\vec{E}=\vec{E_0}e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, we find:

$$-\mu_0\epsilon_0\epsilon(\omega,\vec{k})\omega^2\vec{E}=-k^2\vec{E}\Rightarrow\epsilon(\omega,\vec{k})\omega^2=c^2k^2$$

For frequencies less than $\bar{\omega}_p^2$, ϵ becomes negative, and since ω^2 is positive, $k^2<0$, i.e. $k=ik'$ is imaginary. For the spatial variation of the electric field inside the plasma, this implies $E=E_0e^{iik'r}=E_0e^{-k'r}$, i.e. an exponential damping of the field amplitude. Since $\bar{\omega}_p$ typically lies in the UV-range or above, the critical distance for damping should be of the order $\sim$ 3-400 nm or less.

It is also possible to address this directly through the complex refractive index, $n=N+i\kappa=\sqrt{\epsilon}$, where N represents the real refractive index, whereas the imaginary part reflects losses/absorption. $n=v/c=\omega/(kc)\Rightarrow k=\omega/(c\sqrt{\epsilon})$, or in other words negative ϵ implies imaginary k and n, (i.e. absorption).

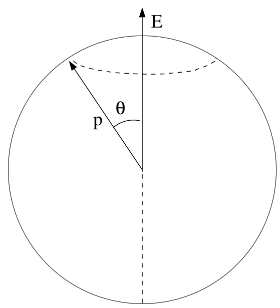
Na has valency of 1, so in the metallic state each atom contributes with 1 electron to the free electron gas. With two Na atoms per u.c., the number density of free electrons will be $n = 2/a^3 = 1.68 \cdot 10^{28}$ m$^-3$. Thus, $\lambda = \frac{2\pi c}{\bar{\omega}_p} = \frac{2\pi\sqrt{\epsilon_{ic}(\infty)m_e}}{e\sqrt{n\mu_0}} = 271$ nm.

Problem 2 - Polarisation

a) The average value of p along the field direction is:

$$< p\cos\theta >= \frac{\int p\cos\theta f(\theta)d\Omega}{\int f(\theta)d\Omega}$$

The ensembly of random dipole orientations with respect to the field can be represented by a spherical sufrace with radius p (see fig). For any given orientation θ, there exists a continuum of possible alignments of the component orthogonal to the field, represented by the circular ring shown in the figure by the half dashed line in the horisontal plane. The relative size of this continuum is represented by the integrand $d\Omega = 2\pi sin\theta d\theta$ (see e.g. analogy with lecture notes for the Lorentz field).



Hence,

$$< p\cos\theta >= \frac{\int_0^\pi p\cos\theta \exp\left(\frac{pE\cos\theta}{k_BT}\right) 2\pi\sin\theta d\theta}{\int_0^\pi \exp\left(\frac{pE\cos\theta}{k_BT}\right) 2\pi\sin\theta d\theta} = \frac{p\int_0^\pi \cos\theta\exp\left(\beta\cos\theta\right)\sin\theta d\theta}{\int_0^\pi \exp\left(\beta\cos\theta\right)\sin\theta d\theta} = \frac{p\int_{-1}^1 xe^{\beta x}dx}{\int_{-1}^1 e^{\beta x}dx}$$

$$= \frac{p\left[\beta^{-2}e^{\beta x}(\beta x - 1)\right]_{-1}^1}{\left[\beta^{-1}e^{\beta x}\right]_{-1}^1} = \frac{\frac{p}{\beta^2}\left(e^{\beta}(\beta-1) + e^{-\beta}(\beta+1)\right)}{\frac{e^\beta - e^{-\beta}}{\beta}} = \frac{p}{\beta}\left(\beta\frac{e^\beta + e^{-\beta}}{e^\beta - e^{-\beta}} - \frac{e^\beta - e^{-\beta}}{e^\beta - e^{-\beta}}\right) = \frac{p}{\beta}\left(\beta\coth\beta - 1\right),$$

with $\beta = \frac{pE}{k_BT}$.

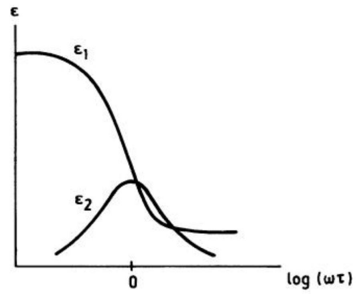
Thus the total polarisation becomes:

$$P = \frac{N < p\cos\theta >}{V} = \frac{nk_BT}{E}\left(\frac{pE}{k_BT}\coth\frac{pE}{k_BT} - 1\right).$$

For weak fields, i.e. $E << k_BT \Rightarrow \beta << 1$, $\coth\beta = \frac{1}{\beta} + \frac{\beta}{3} - \frac{\beta^3}{45} + ... \simeq \frac{k_BT}{pE} + \frac{pE}{3k_BT}$. The orientational polarisability per dipole becomes

$$\alpha = \frac{P}{En} \simeq \frac{nk_BT}{nE^2}\left(\frac{pE}{k_BT}\left[\frac{k_BT}{pE} + \frac{pE}{3k_BT}\right] - 1\right) = \frac{k_BT}{E^2}\left(1 + \frac{p^2E^2}{3(k_BT)^2} - 1\right) = \frac{p^2}{3k_BT} \propto T^{-1}$$

b) Sketch:



At low frequencies, i.e. $\omega << 1/\tau$, the dipoles will reorient constantly with the field alterations, and accordingly the real part of the dielectric response function will be relatively high. As $\omega \to 1/\tau$, however, the response of the oscillating dipoles will be reduced/damped due to an increasing loss to the characteristic "friction", or dipole-molecule interactions, and accordingly ϵ_2 should rise, while ϵ_1 falls off. The exact frequency $\omega = 1/\tau$ represents an extremum (resonance) in terms of heat dissipation into the system, and accordingly ϵ_2 should reach its maximum. At frequencies $\omega > 1/\tau$, the frictional loss fades off again, but at the same time the ability for the dipoles to flip with the increasing field oscillation freqeuncies is reduced, and the real part of the dielectric function will approach a constant high frequency value.

Problem 3 - Superconductivity

a) Bulk s.c. of type I: No fields or gradients (Meissner effect), i.e $\vec{A} = 0, \psi(\vec{r}) = \psi =$ constant. We minimize the free energy function with respect to $|\psi|^2 = n_{cp}$, to find the equilibrium concentration:

$$\frac{\partial f_{sc}}{\partial n_{cp}} = \gamma(T - T_c) + \beta(T) n_{cp} = 0 \Rightarrow (n_{cp})_{eq} = \gamma(T_c - T)/\beta,$$

i.e. $(n_{cp})_{eq} > 0$ when $T < T_c$ and $\beta > 0$.

The critical field is defined by the stabilisation energy of the superconductive state with respect to the normal state, where the energy density of the critical field should correspond to the energy difference between the two states, i.e:

$$\frac{\mu_0}{2} H_c^2(T) = f_n(T) - f_{sc}(T,\psi) = \gamma(T_c - T)(n_{cp})_{eq} - \frac{1}{2}\beta(T)(n_{cp})_{eq}^2 = \frac{\gamma^2(T_c - T)^2}{\beta} - \frac{\gamma^2(T_c - T)^2}{2\beta} = \frac{\gamma^2(T_c - T)^2}{2\beta}$$

$$\Rightarrow H_c(T) = \frac{\gamma(T_c - T)}{\sqrt{\mu_0 \beta}}$$

b) We start with the cooper pair current density:

$$\vec{j}_{cp} = -\frac{2e}{4m_e}\left[\psi^\star i\hbar\nabla\psi + \psi(i\hbar\nabla\psi)^\star\right] - \frac{4e^2}{2m_e}\vec{A}\psi^\star\psi = -\frac{i\hbar 2e}{4m_e}\left[\psi^\star\nabla\psi - \psi\nabla\psi^\star\right] - \frac{4e^2}{2m_e}\vec{A}\psi^\star\psi = -\frac{n_{cp}\hbar e}{m}\nabla\theta(\vec{r}) - \frac{n_{cp}2e^2}{m}\vec{A}.$$

For a type I superconductor in equilibrium (bulk), no gradients should exist, so $\nabla\theta(\vec{r}) = 0$, and $\vec{j}_{cp}$ reduces to

$$\vec{j}_{cp} = -\frac{n_{cp}2e^2}{m}\vec{A}.$$

Maxwells relations:

$$\nabla \times \vec{B} = \mu_0(\nabla \times \vec{H} + \nabla \times \vec{M}) = \mu_0 \vec{j} + \mu_0 \frac{\partial \vec{D}}{\partial t} = \mu_0 \vec{j}$$

(assuming no Amperian and no displacement currents).
Substituting with $\vec{j}_{cp}$ from the expression derived above, gives:

$$\nabla \times \vec{B} = \frac{2e^2 n_{cp} \mu_0}{m_e} \vec{A}$$

We take the curl on both sides to arrive at:

$$\nabla \times \nabla \times \vec{B} = \nabla \cdot \left(\nabla \cdot \vec{B}\right) - \nabla^2 \vec{B} = -\nabla^2 \vec{B} = -\frac{2e^2 n_{cp} \mu_0}{m_e} \nabla \times \vec{A} = -\frac{2e^2 n_{cp}}{m_e \epsilon_0 c^2} \vec{B}$$

.

Thus, $\nabla^2 \vec{B} = \vec{B}/\lambda_L(T)^2$, with $\lambda_L(T) = \sqrt{\frac{m_e \epsilon_0 c^2 \beta}{2e^2 \gamma (T_c - T)}}$, where $(n_{cp})_{eq}$ from a) has been introduced.

λ_L is the so-called London penetration depth, and describes the depth (from the sc surface) where magnetic field penetration is partial, which also coincides with the skin-depth in which the super conductive current may flow. It represents a region where the Meissner effect can be said to be only partial.

c) In the type II superconductor transition regions we assume no fields, but a partial condensation of Cooper pairs, $n_{cp}(r)$. Minimation of the free energy density against the order parameter gives:

$$\frac{\partial f_{sc}}{\partial \psi^\star(r)} = \alpha(T)\psi(r) + \beta n_{cp}(r)\psi(r) - \frac{\hbar^2}{4m_e} \frac{\partial \psi(r)}{\partial r} = 0$$

$$\Rightarrow -\frac{\hbar^2}{4m_e} \frac{\partial \psi(r)}{\partial r} = -\left(\alpha(T) + \beta n_{cp}(r)\right)\psi(r)$$

We divide both sides by $\alpha(T) = -\beta (n_{cp})_{eq}$, i.e. the bulk solution deep in the sc region(see a)), and obtain:

$$-\frac{\hbar^2}{4m_e \gamma (T - T_c)} \frac{\partial \psi(r)}{\partial r} = \left(\frac{n_{cp}(r)}{(n_{cp})_{eq}} - 1\right)\psi(r)$$

The equation represents a modulation of the order parameter $\psi(r)$ as one enters into the transition region. From dimensional analysis it is evident that $\frac{\hbar^2}{4m_e \gamma (T_c - T)}$ must represent a squared length which may be taken as the critical length over which the modulation extends, i.e. the so-called coherence length

$$\xi(T) = \frac{\hbar}{2\sqrt{m_e \gamma (T_c - T)}}$$

Problem 4 - Magnetism

a) Gd^{3+}: $4f^7 5s^2 p^6$, which leaves 7 electrons in the 4f shell, i.e. half-filled $\Rightarrow S = 7/2, L = 3 + 2 + 1 + 0 - 1 - 2 - 3 = 0, J = S = 7/2$.
V^{2+}: $3d^2$. 2 electrons in 3d $\rightarrow S = 1, L = 2 + 1 = 3, J = |L - S| = 2$.

b) Weak fields, $\mu_0\mu_b g(JLS)JH << k_BT$, gives $x << 1$ and $\coth(x) \simeq 1/x + x/3$. Thus, the Brillouin function simplifies to:

$$B_J(x) \simeq \frac{2J+1}{2J}\left(\frac{2J}{2J+1}\frac{1}{x} + \frac{2J+1}{2J}\frac{x}{3}\right) - \frac{1}{2J}\left(\frac{2J}{x} + \frac{x}{6J}\right) = \frac{x}{12J^2}(4J^2+4J) = \frac{J+1}{3J}x$$

Employing the Curie-Brillouin law gives:

$$M = \frac{n\mu_B g(JLS)J(J+1)\mu_B g(JLS)J\mu_0 H}{3Jk_BT} = \frac{n\mu_B^2 g(JLS)^2 J(J+1)\mu_0 H}{3k_BT} = \frac{np^2\mu_B^2\mu_0 H}{3k_BT},$$

and the paramagnetic suceptibility becomes

$$\chi = \frac{\partial M}{\partial H} = \frac{np^2\mu_B^2\mu_0}{3k_BT} = \frac{C}{T},$$

i.e. $C = \frac{np^2\mu_B^2\mu_0}{3k_B}$.

For solids of the two ions from a), and with the given densities, we find:
Gd^{3+}: $n = 2.62\cdot 10^{28}\text{m}^{-3}$, $J = S = 7/2, g(JLS) = \frac{3}{2} + \frac{S(S+1)-0}{2S(S+1)} = 2, p = 2\sqrt{7/2\cdot 9/2} = \sqrt{63} \Rightarrow C \simeq 4.31$ K

V^{2+}: $n = 3.59\cdot 10^{28}\text{m}^{-3}$. Here, we should take into account L-quenching in the solid state, thus: $J = S = 1, L = 0, g(JLS) = 2, p = 2\sqrt{2} = \sqrt{8} \Rightarrow C \simeq 0.75$ K

c) Below T_c: Weiss molecular mean-field direct exchange, $H_{eff} = H_{ext} + \lambda M$ employed with Curie-Brillouin law, directly gives the magentisation in the ferromagnetic phase as

$$M = ng(JLS)\mu_B J B_J\left(\frac{g(JLS)J\mu_0\mu_B H_{eff}}{k_BT}\right) = 7n\mu_B B_{7/2}\left(\frac{7\mu_0\mu_B(H_{ext}+\lambda M)}{k_BT}\right)$$

Above T_c: Applying the Weiss-model, and assuming weak magentisation so that Curies law applies, we find

$$\chi = \frac{\partial M}{\partial H} \simeq \frac{M}{H} = \frac{M}{H_{ext}+\lambda M} = \frac{C}{T} \Rightarrow MT = CH_{ext} + C\lambda M \Rightarrow M = \frac{C}{T - C\lambda}H_{ext}$$

From which

$$\chi = \frac{M}{H_{eff}} \simeq \frac{M}{H_{ext}} = \frac{C}{T - C\lambda} = \frac{C}{T - T_c},$$

i.e. Curie-Weiss law.

Thus, the magnetisation becomes:

$$M = \frac{C}{T - T_c}H$$

Since the Gd^{3+} solid has a paramagnetic-ferromagnetic phase transition at 289 K, the ratio $\chi(300\text{K})/\chi(290\text{K}) = \frac{1}{11}$, and from the CW-law we also find: $\lambda = T_C/C = 289/4.31 \simeq 67$

d) With $J = S = 1$, $J_Z = S_Z = 1, 0, -1$. The energy levels of these three spin states are:

$$E_1 = 2\mu_B B$$

$$E_0 = 0$$

$$E_{-1} = -2\mu_B B$$

From the Boltzman statistics, we find the relative populations of the states to be

$$\frac{N_1}{N} = \frac{e^{-2\mu_B B/k_B T}}{e^{-2\mu_B B/k_B T} + e^{2\mu_B B/k_B T} + 1}$$

$$\frac{N_0}{N} = \frac{1}{e^{-2\mu_B B/k_B T} + e^{2\mu_B B/k_B T} + 1}$$

$$\frac{N_1}{N} = \frac{e^{2\mu_B B/k_B T}}{e^{-2\mu_B B/k_B T} + e^{2\mu_B B/k_B T} + 1}$$

The magentisation of the system

$$M = \sum_{-J}^{J} M_{J_z}(B) N_{J_Z} = \frac{\mu_B}{V}\left(2N_{-1} - 2N_1 + 0N_0\right) = \left(2\frac{e^{2\mu_B B/k_B T}}{N} - 2\frac{e^{-2\mu_B B/k_B T}}{N}\right) n\mu_B$$

$$= \frac{2\left(e^{2\mu_B B/k_B T} - e^{-2\mu_B B/k_B T}\right)}{e^{2\mu_B B/k_B T} + e^{-2\mu_B B/k_B T} + 1} n\mu_B$$

$$= \frac{4\sinh\left(\frac{2\mu_B B}{k_B T}\right)}{2\cosh\left(\frac{2\mu_B B}{k_B T}\right) + 1} n\mu_B$$

The paramagnetic nature of the system is not really altered by the normal-superconductive transition, yet the paramagnetic response vanishes at the transition temperature as the external magnetic field may not any more penetrate into the material, due to the Meissner effect. What happens at the transition is that (the cooper pairs formed in) the free electron gas suddenly starts setting up diamagnetically driven surface currents to block the field penetration out of the material. As the external field is prevented from penetrating into the material, it will be as if there is no field present, i.e. the moments return to a random orientation. For the sake of completeness, please note that a weak contribution from the normal state free electrons to the paramagnetic response has been neglected in the model for M calculated above.