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Exam in TFY4275/FY8907 CLASSICAL TRANSPORT THEORY

May 28, 2014
09:00–13:00

Allowed help: Alternativ **C**
Authorized calculator and mathematical formula book

This problem set consists of 6 pages, plus an Appendix of one page.

This exam consists of three problems each containing several sub-problems. Each of the sub-problems will be given approximately equal weight during grading (if nothing else is said to indicate otherwise). It is estimated that problem three is the most time consuming to answer. Problem two and three may seem long since that have a lot of text, but there is actually not so much time consuming calculations required to answer them. Note also that it may be possible to answer later sub-problems *even* if you were not able to answer correctly all previous sub-problems.

I (or a substitute) will be available for questions related to the problems themselves (though not the answers!). The first round (of two), I plan to do around 10am, and the other one, about two hours later.

The problems are given in English only. Should you have any language problems related to the exam set, do not hesitate to ask. For your answers, you are free to use either English or Norwegian.

Good luck to all of you!

Problem 1.
Consider a discrete-time random walk so that the position after N steps is given by

$$x_N = \sum_{n=1}^{N} \xi_n. \tag{1}$$

Here, as for any random walk process, the jumps ξ_n are assumed to be statistically independent random variables. Moreover, we here also assume them to be drawn for the same probability distribution function.

The jump size (or step length) distribution for a single jump, $p(\xi)$, is defined by the characteristic function

$$G(k) = \hat{p}(k) = \exp\left(-\left|k\right|\right). \tag{2}$$

a) Define what is meant by a characteristic function. Determine a mathematical expression for the jump size distribution $p(\xi)$.

b) Derive an expression for the distribution of the position x_N of the walker, $p_N(x_N)$, valid for any number of steps $N > 0$.

c) Determine the behavior of the mean-square displacement $\left\langle x_N^2 \right\rangle$ in the limit $N \gg 1$. What is this class of random walk processes called?

d) Identify the scaling relation that expresses $p_N(x_N)$ in terms of $p(\xi)$ [*i.e.* $p_N(x_N) \propto p(\xi')$ where $\xi' \propto \xi$]. With such a scaling relation, $p_N(x_N)$ is uniquely determined in terms of $p(\xi)$. What are distributions of such properties called, and why?

e) Determine how the width of the distribution $p_N(x_N)$ increases with increasing N.

Problem 2.
This problem is devoted to the so-called Holtsmark distribution that was mentioned in the lectures. It is a specific distribution from the Levy family with $\alpha = 3/2$ and $\beta = 0$. In 1919, Norwegian physicist Johan Peter Holtsmark (1894–1975), who later became a professor at the Norwegian Institute of Technology (now NTNU), proposed this distribution as a model for the fluctuating fields in plasma due to chaotic motion of charged particles. It is also applicable to other types of Coulomb forces, in particular to modeling of gravitating bodies, and thus is important in astrophysics. We will in this problem take the latter physical application and derive the Holtsmark distribution.

A physical realization of a random walk with a broad distribution of displacements arises in the distribution of gravitational fields — the *Holtsmark distribution* — that is generated by a random distribution of matter.

Consider an infinite system of stars that are *randomly distributed* with a uniform density and with no correlations in spatial positions. We want to compute the distribution of the gravitational force acting on a single "test" star that we take to be located at the origin with no loss of generality. For simplicity, suppose that stars have *equal* masses (the general case of random star masses can also be treated, but the formulae become more cluttered). We are interested in the distribution of the random variable:

$$\mathbf{F} = \sum \mathbf{f}_j \tag{3a}$$

where

$$\mathbf{f}_j = GM\frac{\mathbf{r}_j}{r_j^3} \tag{3b}$$

Here $\mathbf{r}_j$ is the location of the jth star; $\mathbf{f}_j$ is the force on the test star; G is Newton's gravitational constant; and M is the mass of each star. We ignore the factor GM henceforth; it can be easily restored in the final results on dimensional grounds.

We may interpret (3) as an infinite random walk with a broad distribution of (vector) steps $\mathbf{f}_j$. It is convenient to begin with a finite system and then take the thermodynamic limit. Specifically, consider a sphere of radius R with the center at the origin. The number of stars in this sphere is close to $N = nV$, where n is the density of stars and $V = (4/3)\pi R^3$. We first want to determine the finite sum

$$\mathbf{F}_N = \sum_{j=1}^{N} \mathbf{f}_j. \tag{4}$$

a) Show that the probability distribution of $\mathbf{F}_N$ is given by the N-fold integral

$$p_N(\mathbf{F}_N) = \int \frac{\mathrm{d}^3 r_1}{V}\frac{\mathrm{d}^3 r_2}{V}\cdots\frac{\mathrm{d}^3 r_N}{V}\,\delta\left(\sum_{j=1}^{N}\mathbf{f}_j - \mathbf{F}_N\right). \tag{5}$$

b) Obtain the characteristic function of $p_N(\mathbf{F}_N)$, denoted $G_N(\mathbf{k}_N)$, and show that it can be expressed in terms of

$$1 - \frac{1}{V}\int_{\Omega_R} \mathrm{d}^3 r\Big[1 - \exp\left(\mathrm{i}\mathbf{k}_N\cdot\mathbf{f}\right)\Big], \tag{6}$$

where Ω_R is a spherical volume of radius R centered at the origin, and $\mathbf{f} = \mathbf{r}/r^3$. Moreover, recall that we for simplicity have put $GM = 1$.

Take the thermodynamic limit of $G_N(\mathbf{k}_N)$, that is, take the limits $N\to\infty$ and $V\to\infty$, but so that $N/V = n$ remains *finite*, *non-zero* and *constant*.

c) Demonstrate that in the thermodynamic limit, one obtains the characteristic function $G_N(\mathbf{k}_N)\to G(\mathbf{k})$ (and $\mathbf{F}_N\to\mathbf{F}$)

$$G(\mathbf{k}) = \exp\left[-n\int \mathrm{d}^3 r\left(1 - e^{\mathrm{i}\mathbf{k}\cdot\mathbf{f}}\right)\right] \tag{7}$$

where the integral in Eq. (7) is over all of space.

The final force distribution (in the thermodynamic limit) — the Holtsmark distribution — is therefore given by

$$p(\mathbf{F}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3 k\,\exp\left[-\mathrm{i}\mathbf{k}\cdot\mathbf{F} - n\Phi(\mathbf{k})\right], \tag{8a}$$

where we have used the shorthand notation

$$\Phi(\mathbf{k}) = \int \mathrm{d}^3 r \left(1 - e^{\mathrm{i}\mathbf{k}\cdot\mathbf{f}}\right), \qquad \mathbf{f} = \frac{\mathbf{r}}{r^3}. \tag{8b}$$

To obtain explicit results for $p(\mathbf{F})$ we must now compute the integral in Eq. (8b) and then invert the Fourier transform (8a). However, we can determine the dependence of $\Phi(\mathbf{k})$ on $\mathbf{k}$ without calculation.

d) Demonstrate that

$$\Phi(\mathbf{k}) = ak^{3/2} \tag{9}$$

where a is some constant [Hint : show first that $\Phi(\mathbf{k}) = \Phi(k)$].

Together, Eqs. (8) and (9) define the Holtsmark distribution of gravitational fields due to a random distribution of matter. A tedious, but in principle straight forward calculation that you are not asked here to perform, shows that $a = (4/15)(2\pi)^{3/2}$.

Problem 3.

In this problem, we will be concerned with the three dimensional *Lorentz gas model* for which non-interacting classical particles (electrons) move among randomly distributed immobile hard spheres (atoms) of radii a. The electrons do not interact with each other and are elastically scattered by the hard spheres. Consequently, only the direction of the electron velocity $\mathbf{v}$, and not its magnitude, changes in a collision.

For the Lorentz gas model the Boltzmann equation takes the (linear) form (as shown in the lectures) [1]

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}\right) f(\mathbf{r},\mathbf{v},t) = \frac{1}{\tau}\Big\{ [\mathbb{P}f]\,(\mathbf{r},\mathbf{v},t) - f(\mathbf{r},\mathbf{v},t)\Big\} \tag{10a}$$

with a projection operator defined via[2]

$$[\mathbb{P}f]\,(\mathbf{r},\mathbf{v},t) = \int \frac{\mathrm{d}\hat{\mathbf{v}}}{4\pi}\, f(\mathbf{r},\mathbf{v},t), \tag{10b}$$

where $f(\mathbf{r},\mathbf{v},t)$ denotes the probability density to find an electron at position $\mathbf{r}$ with velocity $\mathbf{v}$ at time t. In Eq. (10), $1/\tau = v/\ell$ is the collision frequency; $\ell = (n\pi a^2)^{-1}$ is the mean free path (n is the density of hard spheres); and $\hat{\mathbf{v}} = \mathbf{v}/v$ with $v = |\mathbf{v}|$. Equation (10) is often referred to as the (three dimensional) *Boltzmann-Lorentz* equation.

In the lectures we used the Boltzmann-Lorentz equation (10) to obtain the distribution of velocities

$$F(\mathbf{v},t) = \int \mathrm{d}^3 r\, f(\mathbf{r},\mathbf{v},t), \tag{11}$$

and subsequently used the so-called Einstein-Green-Kubo relation[3] to derive an expression for the diffusion constant that is associated with the three dimensional Lorentz gas model.

The purpose of this problem is solve the Boltzmann-Lorentz equation for the (full) probability density $f(\mathbf{r},\mathbf{v},t)$, and to base an alternative derivation of the diffusion constant, D, on it.

[1] Here $\frac{\partial}{\partial \mathbf{r}}$ denotes the nabla operator $\boldsymbol{\nabla}$.

[2] The integration over $\hat{\mathbf{v}}$ means that we integrate over the direction of the unit vector $\hat{\mathbf{v}}$, that is, the solid angle in velocity space.

[3] Derived in the exam set for 2012 for this class.

a) In your own words, explain the meaning of the left and right hand side of Eq. (10a). What is the meaning of the operator $[\mathbb{P}f]\,(\mathbf{r},\mathbf{v},t)$ defined in Eq. (10b), and what is the physical reason for its presence in the three dimensional Boltzmann-Lorentz equation, Eq. (10a).

We will now try to solve the Boltzmann-Lorentz (BL) equation, Eq. (10a) for the probability density $f(\mathbf{r},\mathbf{v},t)$. The linearity of the BL equation suggests using the Laplace transform in time, as we are interested in $t \geq 0$, and the Fourier transform in space, as we consider the entire three-dimensional space $\mathbb{R}^3$. The Fourier-Laplace transform of the function $f(\mathbf{r},\mathbf{v},t)$ is defined as

$$\tilde{\hat{f}}(\mathbf{k},\mathbf{v},s) = \int_0^\infty \mathrm{d}t\, e^{-st} \int_{-\infty}^{\infty} \mathrm{d}^3r\, f(\mathbf{r},\mathbf{v},t)\, e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}. \tag{12}$$

b) Use the Fourier-Laplace transform technique to obtain a mathematical expression for $\tilde{\hat{f}}(\mathbf{k},\mathbf{v},s)$. Show that it can be written in the form

$$\tilde{\hat{f}}(\mathbf{k},\mathbf{v},s) = \frac{\tau^{-1}}{\tau^{-1}+s+\mathrm{i}\mathbf{k}\cdot\mathbf{v}} \left[\mathbb{P}\tilde{\hat{f}}\right](\mathbf{k},\mathbf{v},s) + \phi_0(\mathbf{k},\mathbf{v},s), \tag{13}$$

and identify the function $\phi_0(\mathbf{k},\mathbf{v},s)$. In your derivation, use the initial condition $f(\mathbf{r},\mathbf{v},t=0) = f_0(\mathbf{r},\mathbf{v})$ with $f_0(\mathbf{r},\mathbf{v})$ a known function.

c) (*Double weight*) To complete the solution, one needs an expression for $\mathbb{P}\tilde{\hat{f}}$. Show that the equation satisfied by $\mathbb{P}\tilde{\hat{f}}$ reads[4]

$$\mathbb{P}\tilde{\hat{f}} = \left[\frac{1}{kv\tau}\arctan\left(\frac{kv\tau}{s\tau+1}\right)\right]\mathbb{P}\tilde{\hat{f}} + \mathbb{P}\phi_0, \tag{14}$$

and use it to establish that in the Fourier-Laplace space the probability density $f(\mathbf{r},\mathbf{v},t)$ satisfies, for an arbitrary initial condition, the equation

$$\tilde{\hat{f}}(\mathbf{k},\mathbf{v},s) = \frac{\tau^{-1}}{\tau^{-1}+s+\mathrm{i}\mathbf{k}\cdot\mathbf{v}}\left[1-\frac{1}{kv\tau}\arctan\left(\frac{kv\tau}{s\tau+1}\right)\right]^{-1}[\mathbb{P}\phi_0\,](\mathbf{k},\mathbf{v},s) + \phi_0(\mathbf{k},\mathbf{v},s). \tag{15}$$

Inverting the exact solution (15) to obtain $f(\mathbf{r},\mathbf{v},t)$, is hard even for the isotropic initial condition (like $f(\mathbf{r},\mathbf{v},t=0) = \delta(\mathbf{r})\delta(v-v_0)/(4\pi v_0^2)$). Nevertheless, the Fourier-Laplace transform allows one to extract the diffusion coefficient. In the hydrodynamic regime of *large spatial and temporal scales*, the density

$$\rho(\mathbf{r},t) = \int \mathrm{d}^3v\, f(\mathbf{r},\mathbf{v},t) \sim \mathbb{P}f \tag{16}$$

should evolve according to the ordinary diffusion equation

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} = D\nabla^2\rho(\mathbf{r},t). \tag{17}$$

[4]For not to clutter the notation, we have here suppressed the function arguments $(\mathbf{k},\mathbf{v},s)$.

d) Obtain an expression for the Fourier-Laplace transformed density, $\tilde{\tilde{\rho}}(\mathbf{k}, s)$, under the assumption of an initial condition $\rho(\mathbf{r}, t = 0) = \delta(\mathbf{r})$.

From the generic form of $\tilde{\tilde{\rho}}(\mathbf{k}, s)$ one may extract the diffusion constant, D, by taking the hydrodynamic limit of Eq. (15). It is recalled that in the hydrodynamic limit, $t \gg \tau$.

e) In the hydrodynamic limit argue why $s \sim t^{-1}$ and $k \sim t^{-1/2}$, and together with $t \gg \tau$ (hydrodynamic limit), show that in this limit it follows that $\tau^{-1} \gg s$ and $\tau^{-1} \gg kv$. Introduce these results into Eq. (15) and show that it takes the form

$$\tilde{\tilde{f}}(\mathbf{k}, \mathbf{v}, s) = \left[1 - \frac{1}{kv\tau} \arctan\left(\frac{kv\tau}{s\tau + 1}\right)\right]^{-1} [\mathbb{P}\phi_0]\,(\mathbf{k}, \mathbf{v}, s) + \phi_0(\mathbf{k}, \mathbf{v}, s). \qquad (18)$$

Also identify the expression for $\phi_0(\mathbf{k}, \mathbf{v}, s)$ valid in the hydrodynamic limit.

f) Finally deduce the diffusion constant, D, by comparing the scaling forms of $\tilde{\tilde{\rho}}(\mathbf{k}, s)$ and $\tilde{\tilde{f}}(\mathbf{k}, \mathbf{v}, s)$ in the hydrodynamic limit. [Hint expand the expression in the square brackets of Eq. (18).]

Appendix

- $$\int e^{cx} \cos bx \; \mathrm{d}x = \frac{e^{cx}}{c^2 + b^2}(c \cos bx + b \sin bx) \tag{19}$$

- $$\int_0^\infty e^{-ax} \cos bx \, \mathrm{d}x = \frac{a}{a^2 + b^2} \quad (a > 0) \tag{20}$$

- $$\int_{-1}^{1} \frac{\mathrm{d}x}{a + \mathrm{i}bx} = \frac{2}{b} \arctan\left(\frac{b}{a}\right) \quad (a, b > 0) \tag{21}$$