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Exam in TFY4275 Classical Transport Theory
Saturday, May 16, 2015
09:00–13:00

Allowed help: Alternativ C

This problem set consists of 4 pages.

Problem 1

We will in this problem consider *magnetotransport* in the Lorentz model. The Lorentz model consists of a particle colliding elastically with static scatterers. We assume the particle to carry an electrical charge $(-e)$ and a mass m. There is a constant and spatially homogeneous magnetic field $\vec{B} = B\vec{e}_z$ in the z-direction. Hence, between collisions, the charged particle is subjected to the Lorentz force and follows the equation of motion

$$m\dot{\vec{v}} = -e\vec{v} \times \vec{B} \ , \tag{1}$$

where $\dot{\vec{v}}$ is the time derivative of the velocity $\vec{v}$.
We will in the following assume that the system is *two dimensional*. That is, the charged particle is moving in the (x, y) plane which is perpendicular to the magnetic field. The scatterers are also two dimensional so that the charged particle never leaves this plane, even after being scattered.

a) If the charged particle does not collide with any scatterer, it will move in a circle in the (x, y) plane. Show that if the charged particle's speed is $v = |\vec{v}|$, then the radius of the circle is

$$R = \frac{mv}{eB} \ , \tag{2}$$

and the angular frequency, $\omega = 2\pi/T$ where T is the time it takes to complete the circle, is

$$\omega = \frac{eB}{m} \ . \tag{3}$$

b) We assume that the scatterers are disks with radii equal to a. The differential scattering cross section for such disks is

$$\frac{d\sigma(\psi)}{d\psi} = \frac{a}{2} \ \sin\left|\frac{\psi}{2}\right| \ , \tag{4}$$

where ψ is the scattering angle with respect to the velocity vector of charged particle just before the collision. Show that the total cross section is

$$\sigma = 2a \; . \tag{5}$$

We define a dimensionless differential cross section as

$$g(\psi) = \frac{1}{\sigma}\,\frac{d\sigma(\psi)}{d\psi} = \frac{1}{4}\,\sin\left|\frac{\psi}{2}\right| \; . \tag{6}$$

c) We assume a uniform density of scatterers (i.e. number of scatterers per area) equal to n. Show that the mean free path of the charged particle (distance between scattering events) is

$$\Lambda = \frac{1}{n\sigma} \; . \tag{7}$$

Furthermore, show that the mean time between collisions is

$$\tau = \frac{1}{nv\sigma} \; . \tag{8}$$

d) We will in the following work in the *Grad limit* (named after H. Grad) where $n \to \infty$ and $a \to 0$ in such a way that Λ remains constant. Suppose the charged particle is released at a position $\vec{r}_0$ at time $t = 0$ with a velocity $\vec{v}$. Show that the probability that the particle will *not* suffer any collisions with the scatterers is

$$P_0 = e^{-2\pi R/\Lambda} = e^{-T/\tau} \; . \tag{9}$$

Hint: Start by showing that the probability that the particle does not suffer a collision over an infinitesimal distance dl is $(1 - dl/\Lambda)$.

If the charged particle does not hit a scatterer after having completed a full circle, it will *never* collide with the scatterers. Rather, it will for ever be repeating the same circle.

e) We now construct the Boltzmann equation for this problem: the two-dimensional Lorentz model in a perpendicular magnetic field and in the Grad limit. The probability density to find the charged particle with a velocity direction characterized by an angle ϕ and at time t is $f(\phi, t)$. The left hand side of the Boltzmann equation contaning the flow terms may then be written

$$\left[\frac{\partial}{\partial t} + \omega\,\frac{\partial}{\partial \phi}\right]\, f(\phi, t) \; . \tag{10}$$

Argue why this is so.

f) The Boltzmann collision operator stating how $f(\phi, t)$ changes per time due to collisions with new scatterers is given by

$$\mathcal{B}_0\left[f(\phi, t)\right] = \frac{1}{\tau}\int_{-\pi}^{+\pi} d\psi\; g(\psi)\left[f(\phi - \psi, t) - f(\phi, t)\right] \; . \tag{11}$$

Explain the terms in this expression.

g) Suppose now that the charged particle has just hit a scatterer, thereby changing its velocity vector from direction $\phi-\psi$ to ψ. There is then a probability P_0 — see equation (9) — that the charged particle will return after a time $T = 2\pi/\omega$ to *collide once more with the same scatterer.* The new direction of the charged particle after the second collision with the same scatterer will be $\phi+\psi$. Show this. (Hint: Work with a scatterer that has a finite radius $a \ll R$. A simple geometrical argument will do.)

h) We are now in the position to construct the Boltzmann equation for this problem. Suppose the charged particle collides k times with the same scatterer before hitting a new one. The rate at which this happens is given by

$$\mathcal{B}_k\left[f(\phi,t)\right] = \frac{1}{\tau}\, P_0^k \int_{-\pi}^{+\pi} d\psi\, g(\psi)\left[f(\phi-(k+1)\psi, t-kT) - f(\phi-k\psi, t-kT)\right] \ . \qquad (12)$$

Explain the reasoning behind this term.

i) The full — and *exact* — Boltzmann equation for this problem is then

$$\left[\frac{\partial}{\partial t} + \omega\,\frac{\partial}{\partial \phi}\right]\ f(\phi,t) = \sum_{k=0}^{[t/T]} \mathcal{B}_k\left[f(\phi,t)\right] \ , \qquad (13)$$

where $[t/T]$ is the integer part of the ratio t/T (as in 5 is the integer part of 5.2). This equation breaks two assumptions usually seen as essential for constructing Boltzmann equations that can be solved. Which two assumptions are they?

j) Equation (13) can be solved analytically. We will not do this here since the ensuing expressions are longish. However, we will solve it in the limit of weak magnetic field. Show that the assumption $T \gg \tau$ corresponds to this limit.

k) In this weak-field limit, the Boltzmann equation (13) may be approximated by

$$\left[\frac{\partial}{\partial t} + \omega\,\frac{\partial}{\partial \phi}\right]\ f(\phi,t) = \mathcal{B}_0\left[f(\phi,t)\right] \ , \qquad (14)$$

by neglecting terms proportional with P_0 in the exact Boltzmann equation.

By defining the Fourier transform of $f(\phi,t)$ as

$$f_m(t) = \int_{-\pi}^{+\pi} d\phi\ e^{im\phi}\ f(\phi,t) \ , \qquad (15)$$

and the Fourier-Laplace transform as

$$F_m(s) = \int_0^{\infty} ds\ e^{-st}\ f_m(t) \ , \qquad (16)$$

show that the solution of equation (14) is

$$F_m(s) = \frac{\tau(1-4m^2)f_m(0)}{\tau(1-4m^2)(s-im\omega)-4m^2} \ . \qquad (17)$$

Hint:

$$g_m = \int_{-\pi}^{+\pi} d\psi\ e^{im\psi}\ g(\psi) = \frac{1}{1-4m^2} \qquad (18)$$

is the Fourier transform of the dimensionless differential scattering cross section (6). The parameter m takes on integer values only.

l) We now use the Einstein-Green-Kubo formula to determine the diffusion constant D in our system. The Einstein-Green-Kubo formula in two dimensions is

$$D = \frac{1}{2} \int_0^\infty dt \, \langle \vec{v}(t) \cdot \vec{v}(0) \rangle \, . \tag{19}$$

We orient the cartesian coordinate system (x, y) so that the charged particle initially has the velocity pointing along the x-direction corresponding to $\phi(t = 0) = 0$. Hence, $v_x(0) = v\cos(0) = v$ and $v_y(0) = v\sin(0) = 0$. At later times $v_x(t) = v\cos(\phi(t))$. Hence, the diffusion constant is

$$D = \frac{v^2}{2} \int_0^\infty dt \, \langle \cos(\phi(t)) \rangle \, . \tag{20}$$

Show that

$$D = \frac{v^2}{2} \, \mathcal{R}e \, [F_1(0)] \, . \tag{21}$$

by first showing that

$$F_m(s) = \int_0^\infty dt \, e^{-st} \, \langle e^{im\phi(t)} \rangle \, . \tag{22}$$

In equation (21) $\mathcal{R}e$ means real part. Finally, show that

$$D = \frac{3v^2\tau}{8 + 18\pi^2(\tau/T)^2} \, . \tag{23}$$

It should be noted that this expression is *not* a series expansion in the small parameter $\tau/T \propto B$. Rather, the *only* approximation consists in neglecting recollisions with the same scatterer.