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Exam in TFY4275/FY8907 Classical Transport Theory
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Allowed help: Alternativ C

This problem set consists of 3 pages.

Problem 1

We will in this problem analyze the motion of a steel ball rolling on a horizontal sheet of sandpaper using the theory for point processes. The steel ball is assumed to be much larger than the sand grains glued to the surface of the sand paper. When the steel ball is rolling, it hits the sand grains inelastically and kinetic energy is lost. This loss is quantified by the introduction of the *coefficient of restitution* η, where $0 \leq \eta \leq 1$. If the speed of the steel ball before the collision with the sand grain is v_i, then it is $v_{i+1} = \eta v_i$ after the collision. The kinetic energy is

$$E_i = \frac{M}{2}\, v_i^2 \,, \tag{1}$$

and

$$E_{i+1} = \frac{M}{2}\, v_{i+1}^2 = \eta^2 E_i \,, \tag{2}$$

after the collision. M is the mass of the steel ball. Our aim is to calculate the average speed of the ball as a function of time.

As the steel ball rolls, it experiences multiple collisions as it rolls over the sand grains. We will model these collisions as a random, uncorrelated point process.

a) We define the probability that s collisions happen at times $\tau_1, \ldots \tau_s$ to be

$$Q_s(\tau_1, \cdots, \tau_i, \cdots, \tau_j, \cdots, \tau_s) = Q_s(\tau_1, \cdots, \tau_j, \cdots, \tau_i, \cdots, \tau_s) \,. \tag{3}$$

We have here indicated that we have symmetrized the distribution. The normalization condition on the distribution is

$$Q_0 + \sum_{s=1}^{\infty} \frac{1}{s!} \int_{-\infty}^{\infty} d\tau_1 \ldots d\tau_s \; Q_s(\tau_1, \cdots, \tau_s) = 1 \,. \tag{4}$$

Suppose $A = A_s(\tau_1, \cdots, \tau_s)$ is a quantity that depends on the collisions. What is the average $\langle A \rangle$ expressed in terms of Q_s given that it has the same symmetries with respect to τ_i as Q_s?

b) Suppose the steel ball suffers N collisions. Its kinetic energy is then

$$E_N = E_0 \eta^{2N} \ . \tag{5}$$

We introduce the indicator function

$$\chi(\tau) = \begin{cases} 1 & \text{if } 0 \leq \tau \leq t \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{6}$$

Argue that

$$E_s(\tau_1, \cdots, \tau_s) = E_0 \eta^{2 \sum_{\sigma=1}^{s} \chi(\tau_\sigma)} \ , \tag{7}$$

is the proper kinetic energy function describing a ball beginning to roll at time $t = 0$.

c) We now assume that the collisions are independent. Hence, Q_s describes a independent random point process. We then have that

$$Q_s(\tau_1, \cdots, \tau_s) = e^{-\nu} q(\tau_1) \ldots q(\tau_s) \ , \tag{8}$$

and

$$Q_0 = e^{-\nu} \ . \tag{9}$$

Show that

$$\nu = \int_{-\infty}^{\infty} d\tau \ q(\tau) \ . \tag{10}$$

d) We now assume that the collision probability has the form

$$q(\tau) = \begin{cases} \frac{\nu}{2T} & \text{if } -T \leq \tau \leq T \ , \\ 0 & \text{otherwise} \ , \end{cases} \tag{11}$$

letting $T \to \infty$ and $\nu \to \infty$ so that the ratio $\rho = \nu/(2T)$ remains a finite constant. What is the physical interpretation of ρ?

e) Show that the average kinetic energy as a function of time is

$$\langle E \rangle = E_0 e^{(\eta^2 - 1)\rho t} \ . \tag{12}$$

In terms of speed decay, this becomes

$$v_{RMS} \equiv \langle v^2 \rangle^{1/2} = v_0 e^{(\eta^2-1)\rho t/2} \ , \tag{13}$$

where RMS stands for *root mean square.*

f) Rather than using the kinetic energy decay, let us now base ourselves on the decay of the speed directly, i.e., that $v_{i+1} = \eta v_i$. Show that

$$\langle v \rangle = v_0 e^{(\eta-1)\rho t} \ . \tag{14}$$

g) Why is $v_{RMS} \neq \langle v \rangle$?

We see that $\ln(v_{RMS})/\ln(\langle v \rangle) = (\eta+1)/2 = 1-\delta/2$, where we have written $\eta = 1-\delta$. In practice, $\delta \ll 1$ for this system. Hence, the two expressions for the speed are quite close.

h) This theory for the behavior of a steel ball rolling on a horizontal sandpaper contains many physical simplifications. In your mind, which are the two most important ones?

Problem 2

We will in this problem study the *random telegraph process,* or — as it is also known — the *dichotomic Markov process.* The stochastic variable y takes on the values ± 1. The conditional transition probability is

$$P_{1|1}(y,t|y',t') = \frac{1}{2}\left[1+e^{-2\gamma(t-t')}\right]\delta_{y,y'} + \frac{1}{2}\left[1-e^{-2\gamma(t-t')}\right]\delta_{y,-y'} \ , \tag{15}$$

where γ is a parameter setting the time scale.

a) Show that the random telegraph process (15) obeys the Chapman-Kolmogorov equation

$$P_{1|1}(y_3,t_3|y_1,t_1) = \int dy_2\ P_{1|1}(y_3,t_3|y_2,t_2)P_{1|1}(y_2,t_2|y_1,t_1) \ . \tag{16}$$

(Hint: Introduce the notation $[1 \pm \exp(-2\gamma(t_i - t_j))]/2 = (\pm)_{ij}$ and use the two rules $(+)_{ij}(+)_{jk} + (-)_{ij}(-)_{jk} = (+)_{ik}$ and $(+)_{ij}(-)_{jk} + (-)_{ij}(+)_{jk} = (-)_{ik}$. This to avoid long expressions.)

b) Show that the random telegraph process is consistent with the stationary probability distribution

$$P_1(y,t) = \frac{1}{2}\left[\delta_{y,1} + \delta_{y,-1}\right] \ . \tag{17}$$

c) The master equation is

$$\frac{\partial}{\partial t}\ P(y,t) = \int \left[W(y|y')P(y',t) - W(y'|y)P(y,t)\right] dy' \ . \tag{18}$$

Describe the physical contents of this equation.

d) The transition probabilities $W(y|y')$ is related to the conditional probability $P_{1|1}(y,t|y',t')$ through the expression

$$P_{1|1}(y,t|y',t') = \left[1-(t-t')a_0(y)\right]\delta(y-y') + (t-t')W(y|y') + \mathcal{O}(t-t') \ , \tag{19}$$

where $a_0(y) = \int dy'\ W(y'|y)$. We have assumed that $P_{1|1}(y,t|y',t')$ only depends on time differences $t-t'$.

Find the master equation for the random telegraph process.

e) Show that (17) solves the master equation for the random telegraph process.