<u>Problem 1.</u> (Points: 10+10+10+10+10+10= 60)

The Boltzmann-equation for the one-particle distribution function $f(\mathbf{r}, \mathbf{v}, t)$ is given by

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = \int d^3 v_1 \int \; d\Omega \; \sigma(\Omega) \; |\mathbf{v}_1 - \mathbf{v}| \; [f' f_1' - f f_1]$$

The collision term on the right hand side of the equation describes a collision $(\mathbf{v}, \mathbf{v}_1) \to (\mathbf{v}', \mathbf{v}_1')$, with $f = f(\mathbf{r}, \mathbf{v}, t), f_1 = f(\mathbf{r}, \mathbf{v}_1, t)$, and correspondingly for (f', f_1'). Finally, $\sigma(\Omega)$ is the differential scattering cross section for the collision.

a) Define the functional

$$H(t) \equiv \int d^3 r \int d^3 v f \; ln f$$

Assume now that $\mathbf{a} = -\partial U(\mathbf{r})/\partial \mathbf{r}$, where U is some external potential, and show that

$$\frac{dH}{dt} \leq 0$$

Give the physical interpretation of this result.

b) A velocity-average of a quantity $A(\mathbf{v})$ may be obtained from f as

$$\langle A \rangle \equiv \frac{1}{n} \int d^3 v A f$$

where $n = \int d^3 v f$, where n is a number density. Show that we have the following macroscopic equation for any $A(\mathbf{v})$

$$\frac{\partial \left(n\langle A \rangle\right)}{\partial t} + \frac{\partial \left(n\langle A v_j \rangle\right)}{\partial x_j} - a_j \frac{\partial \left(n\langle A \rangle\right)}{\partial v_j} = \int d^3 v \int d^3 v_1 \int \; d\Omega \; \sigma(\Omega) \; |\mathbf{v}_1 - \mathbf{v}| \; [f' f_1' - f f_1] \, A(\mathbf{v})$$

c) Define what is meant by a collision invariant, and show that the right hand side of the above equation vanishes whenever A is a collision invariant.

d) Now let $A = \mathbf{p}$, where $\mathbf{p}$ is the momentum of the particle described by f. Let $\rho = nm$ be the mass-density of the system. Set up the macroscopic equation for $\mathbf{p}$ and give the physical interpretation of it.

e) The hydrodynamic conservation laws for mass and momentum are given by, on their most general form

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) &= \mathbf{F} - \nabla \cdot \mathbb{P}
\end{aligned}$$

where $\mathbb{P}$ is the pressure-tensor of the system, $\mathbf{F}$ is an external force acting on a little fluid element of density ρ, and $\mathbf{u} = \langle \mathbf{v} \rangle$. Express $\mathbb{P}$ in terms of an appropriate velocity-average, as defined above, by comparing the above equations with the result you found in **d**).

f) Consider the system close to equilibrium, so that we may use the following Ansatz for f

$$f(\mathbf{r}, \mathbf{v}, t) = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{m}{2k_B T} (\mathbf{v} - \mathbf{u})^2 \right)$$

where k_B is Boltzmann's constant and T is temperature, and $(n, T, \mathbf{u})$ could depend on $(\mathbf{r}, t)$. Calculate $\mathbb{P}$ for this case, and from this find expressions for the hydrostatic pressure p, the shear viscosity η, and bulk-viscosity ζ of the system.

<u>Problem 2.</u> (Points: 10+10+10+10=40)

The Master equation for a discrete, continuous time, stochastic process is given by

$$\dot{P}_n(t) = \sum_{n'} \left[\omega_{n,n'} P_{n'} - \omega_{n',n} P_n\right] \equiv \mathbb{W} P_n$$

Here, $P_n(t)$ is the probability distribution of the stochastic process, and $\omega_{n,n'}$ is a transition rate from state n' to state n. The matrix $\mathbb{W}$ has eigenfunctions $\Phi_\lambda(n)$ and eigenvalues λ given by

$$\mathbb{W}\Phi_\lambda(n) = -\lambda \Phi_\lambda(n)$$

We define an inner product as follows:

$$(\Phi, \Psi) \equiv \sum_n \frac{\Phi(n)\Psi(n)}{P_n^0}$$

Here, P_n^0 is the equilibrium solution to the Master equation, which we consider to be known.

a) Assume detailed balance to hold, and use this to show that $(\Phi, \mathbb{W}\Psi) = (\Psi, \mathbb{W}\Phi)$ and that $\lambda \geq 0$. Write down a general solution to the Master equation using the eigenvalues and eigenfunctions of $\mathbb{W}$. In particular, exhibit the solution at $t \to \infty$.

b) A continuous time asymmetric random walk is defined by the transition rate $\omega_{n,n'}$

$$\omega_{n,n'} = \alpha\ \delta_{n+1,n'} + \beta\ \delta_{n-1,n'}; \quad (\alpha, \beta) \geq 0$$

Give a physical interpretation of the constants α and β and set up the Master equation for this problem. Give the balance condition and the detailed balance condition for stationary states.

c) Introduce the generating functional $F(z,t)$ for $P_n(t)$, defined by

$$F(z,t) \equiv \sum_n z^n P_n(t)$$

Find the differential equation for $F(z,t)$ with initial condition obtained from $P_n(0) = \delta_{n,0}$, solve this equation and find $P_n(t)$.

d) Define the drift-velocity v_{drift} and diffusion constant D for this system, by

$$\begin{aligned} \langle n \rangle &= v_{\text{drift}} t \\ \langle (n - \langle n \rangle)^2 \rangle &= Dt \end{aligned}$$

Here, an average is defined by

$$\langle f(n) \rangle(t) \equiv \sum_n f(n) P_n(t)$$

Calculate v_{drift} and D.

Formulae that may be useful:

$$
\begin{aligned}
e^x &= \sum_{k=0}^{\infty} \frac{1}{k!}\, x^k \\
(a+b)^N &= \sum_{n=0}^{N} \binom{N}{k}\, a^{N-n}\, b^n
\end{aligned}
$$

$$\int_0^\infty dx\ x^{2n}\ e^{-\alpha x^2} = \left(-\frac{d}{d\alpha}\right)^n \sqrt{\frac{\pi}{\alpha}}$$

Constitutive relation for the hydrodynamic pressure tensor to linear order in gradients:

$$\mathbb{P}_{ij} = p\ \delta_{ij} - 2\eta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \zeta \delta_{ij} \nabla \cdot \mathbf{u}$$

The Wiener-Khinchin theorem for the power spectrum of a stochastic process $Y(t)$:

$$G(\omega) = \frac{2}{\pi} \int_0^\infty d\tau\ \cos(\omega\tau)\ \kappa(\tau)$$

where $\kappa(\tau) = \langle Y(t+\tau) Y(t) \rangle$. The Langevin-equation is given by

$$\dot{\mathbf{v}} + \gamma \mathbf{v} = A(t)$$

with $\langle A(t) \rangle = 0$, $\langle A(t+\tau) A(t) \rangle = \Gamma \delta(\tau)$.

The probability-distribution of a one-component stochastic process $V(t)$ governed by the Langevin-equation is given by, with $V(0) = V_0$

$$P(V,t) = \sqrt{\frac{\gamma}{\pi\Gamma(1-e^{-2\gamma t})}}\ \exp\left(-\frac{\gamma(V - V_0 e^{-\gamma t})}{\Gamma(1-e^{-2\gamma t})}\right)$$