

# TFY 4275 : Classical Transport Theory

Solution (sketch) V2008

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## Problem 1

a) Ordinary diffusion :  $\langle x^2(t) \rangle \sim 2Dt$   
(linear in time)

b) Anomalous diffusion corresponds to  $\langle x^2(t) \rangle$   
scales non-linearly with time.

- sub-diffusion :  $\langle x^2 \rangle$  increases slower than  $t$   
Can be caused by long waiting times  
Non-Markovian

- super-diffusion :  $\langle x^2 \rangle$  increases faster than  $t$   
May be caused by long jumps.  
Markovian

$$\square \quad \langle |x|^\delta \rangle = \int_{-\infty}^{\infty} dx |x|^\delta \mathcal{L}_x(x) = 2 \int_0^{\infty} dx x^\delta \mathcal{L}_x(x)$$

The integrand of this integral scales like

$$x^{\delta - (\alpha + 1)}$$

For the integral to converge one must have

$$\delta - (\alpha + 1) < -1$$

that is, if  $\delta < \alpha$ .

If  $L_\alpha(x) \rightarrow p(x) \sim x^{-(\alpha+1)}$  (and otherwise well behaved) nothing will change, and the above condition still applies.  $\langle |x|^\delta \rangle$  will still be finite when  $\alpha \geq 2$  if  $\delta < \alpha$ .

d) From the previous sub-problem, it follows that  $\langle x^2 \rangle$  and  $\langle t \rangle$  are finite if

$$\begin{aligned} \mu &> 2 \\ \delta &> 1 \end{aligned}$$

Hence:

- 1) Ordinary diff
- 2) sub-diffusion
- 3) can be anything
- 4) ordinary diffusion
- 5) super-diffusion
- 6) super-diffusion

## Problem 2

a) Jump size prob

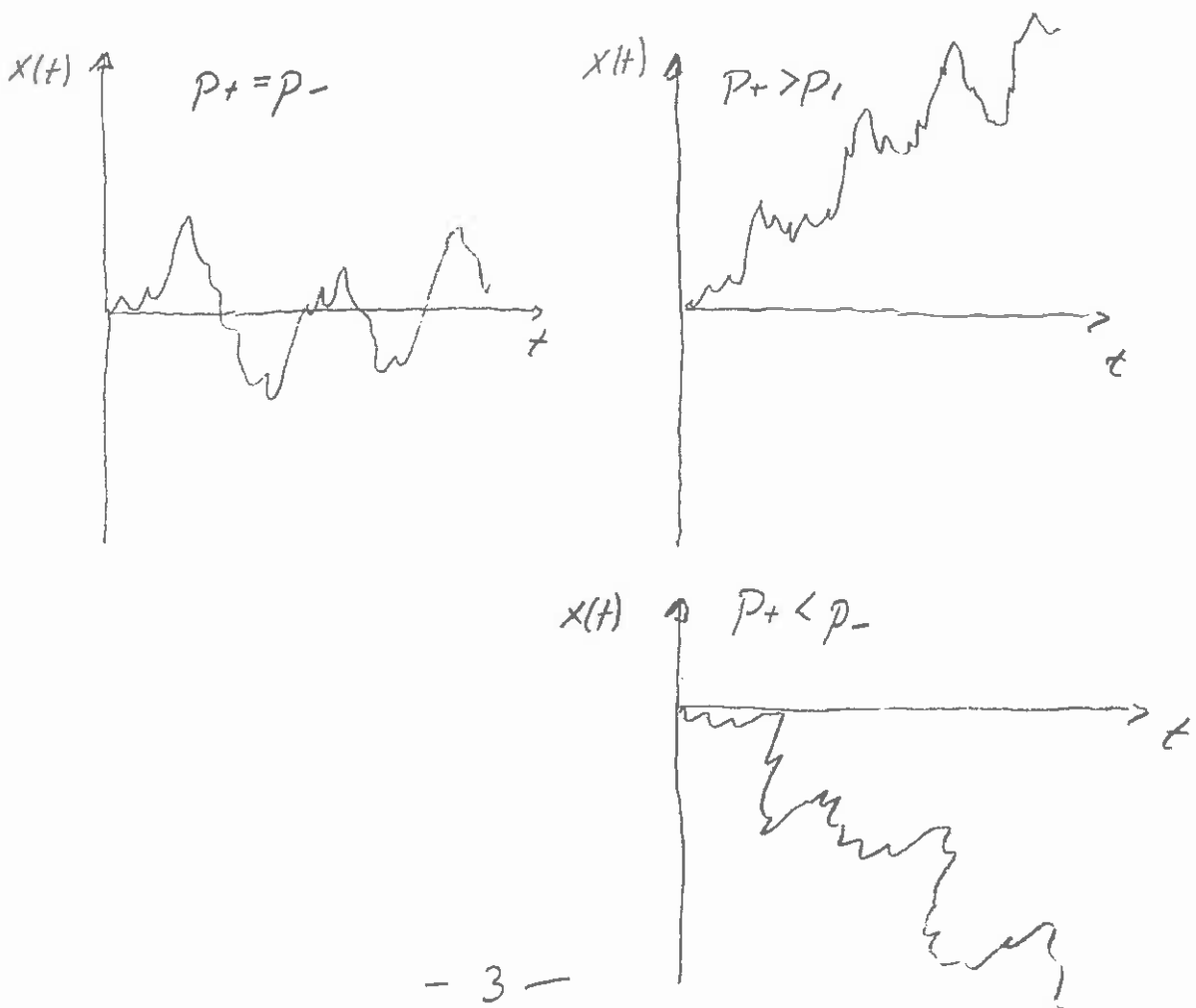
$$p_i(\xi) = p_+ \delta(\xi - \Delta x) + p_- \delta(\xi + \Delta x) + (1 - p_+ - p_-) \delta(\xi)$$

$$\begin{aligned} \hat{p}_i(k) &= p_+ e^{-ik\Delta x} + p_- e^{ik\Delta x} + 1 - p_+ - p_- \\ &= 1 + p_+(e^{-ik\Delta x} - 1) + p_-(e^{ik\Delta x} - 1) \end{aligned}$$

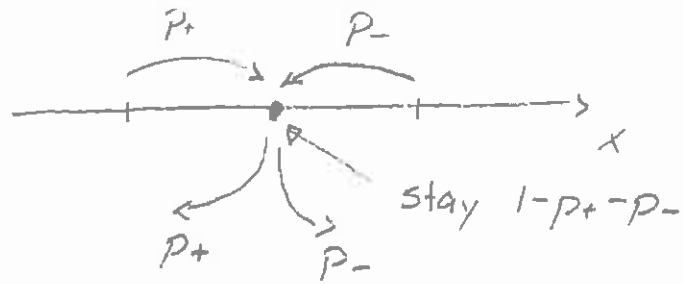
b)  $\langle \xi \rangle = \int d\xi \xi p_i(\xi) = (p_+ - p_-) \Delta x$

Average drift velocity:

$$v_{\text{drift}} = \frac{\langle \xi \rangle}{\Delta t} = (p_+ - p_-) \frac{\Delta x}{\Delta t} \leq \frac{\Delta x}{\Delta t}$$



c)



$$\begin{aligned} P(x, t + \Delta t) &= \underbrace{+ p_+ P(x - \Delta x, t) + p_- P(x + \Delta x, t)}_{\text{in-flow}} \\ &\quad - \underbrace{p_+ P(x, t) - p_- P(x, t)}_{\text{out-flow}} \\ &\quad + P(x, t) \\ &= P(x, t) + p_+ [P(x - \Delta x, t) - P(x, t)] \\ &\quad + p_- [P(x + \Delta x, t) - P(x, t)] \end{aligned}$$

d)

$$P(x, t + \Delta t) \simeq P(x, t) + \Delta t \partial_t P(x, t) + \dots$$

To first order in  $\Delta t$  we have

$$\begin{aligned} \partial_t P(x, t) &= r_+ [P(x - \Delta x, t) - P(x, t)] \\ &\quad + r_- [P(x + \Delta x, t) - P(x, t)], \quad r_{\pm} = \frac{p_{\pm}}{\Delta t} \end{aligned}$$

This is the Master eq. for the process.

$$\square f(x \pm \Delta x, t) = f(x, t) \pm \Delta x \partial_x f(x, t) + \frac{\Delta x^2}{2!} \partial_x^2 f(x, t) + \dots$$

$$\begin{aligned} \partial_t f(x, t) &= r_+ \left[ f(x, t) - \Delta x \partial_x f(x, t) + \frac{\Delta x^2}{2!} \partial_x^2 f(x, t) + \dots \right. \\ &\quad \left. - f(x, t) \right] \\ &\quad + r_- \left[ f(x, t) + \Delta x \partial_x f(x, t) + \frac{\Delta x^2}{2!} \partial_x^2 f(x, t) + \dots \right. \\ &\quad \left. - f(x, t) \right] \\ &= -(r_+ - r_-) \Delta x \partial_x f(x, t) + (r_+ + r_-) \frac{\Delta x^2}{2!} \partial_x^2 f(x, t) \\ &\quad + \dots \end{aligned}$$

Hence to second order in  $\Delta x$  the given eq. follows with

$$v = (r_+ - r_-) \Delta x = (p_+ - p_-) \frac{\Delta x}{\Delta t}$$

$$D = (r_+ + r_-) \frac{\Delta x^2}{2!} = (p_+ + p_-) \frac{\Delta x^2}{2 \Delta t}$$

This is the Fokker-Planck eq.  
Specially when  $p_+ = p_- = \frac{1}{2}$

$$v = 0$$

$$D = \frac{\Delta x^2}{2 \Delta t}$$

as we saw in the lectures in the symmetric case.

f] The  $v$ -term is a drift term (diffusion-advection eq.). This term can be transformed "away" by going to a movable reference frame ( $x-vt$ ). In this frame one has just the regular diffusion eq. Hence, the solution is the one given.

$$\begin{aligned}
 \text{g] } \langle x(t) \rangle &= \int_{-\infty}^{\infty} dx \, x f(x,t) \\
 &= \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} du \, (u+x_0+vt) e^{-u^2/4Dt} \\
 &= \frac{1}{\sqrt{4\pi Dt}} (x_0+vt) \sqrt{\pi 4Dt} \\
 &= x_0+vt
 \end{aligned}$$

$$\begin{aligned}
 \langle x^2(t) \rangle &= \int_{-\infty}^{\infty} dx \, x^2 f(x,t) \\
 &= \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} du \, \frac{(u+x_0+vt)^2 e^{-u^2/4Dt}}{u^2 + 2u(x_0+vt) + (x_0+vt)^2} \\
 &= \frac{1}{\sqrt{4\pi Dt}} \left[ \frac{4Dt}{2} + (x_0+vt)^2 \right] \sqrt{\pi 4Dt} \\
 &= 2Dt + (x_0+vt)^2
 \end{aligned}$$

$$\sigma_x(t) = [\langle x^2(t) \rangle - \langle x \rangle^2]^{1/2} = \sqrt{2Dt}$$

Note: The change in the details of the RW does not change the scaling!

Alternatively these results may be obtained from the FP equation. Multiply the FP eq. by  $x$  and integrate over all space:

$$\int_{-\infty}^{\infty} dx \, x \partial_t f(x,t) = -\gamma \int_{-\infty}^{\infty} dx \, x \partial_x f(x,t) + D \int_{-\infty}^{\infty} dx \, x \partial_x^2 f(x,t)$$

$$\begin{aligned} \partial_t \langle x(t) \rangle &= -\gamma [x f(x,t)] \Big|_{x=-\infty}^{\infty} + \gamma \int_{-\infty}^{\infty} dx \, f(x,t) \\ &\quad + D [x \partial_x f(x,t)] \Big|_{x=-\infty}^{\infty} - D \int_{-\infty}^{\infty} dx \, \partial_x f(x,t) \\ &= \gamma \end{aligned}$$

The solution of this eq. is as before

$$\langle x(t) \rangle = \gamma t + x_0$$

where we have used that  $\langle x(t=0) \rangle = x_0$ .

In a similar way, one may derive an eq. for  $\langle x^2(t) \rangle$  by multiplying the FP eq. by  $x^2$  and integrating over all space.

### Problem 3

a) Shortly after  $t=0$  the positive ions will start moving towards the wall in order to try to "cancel" the effect of the negative potential.

For long times the concentrations  $C_{\pm}(x)$  will reach stationary solutions.

Far away from the wall one should have,  $C_{\pm}(x=\infty) = C_0$ , i.e. the initial concentrations.

b) The Fokker-Planck eq:

$$\partial_t C_{\pm}(x) = -\partial_x [\mu_{\pm} F_{\pm}(x) C_{\pm}(x)] + D_{\pm} \partial_x^2 C_{\pm}(x)$$

stationary solution when  $\partial_t C_{\pm}(x) = 0$ :

$$\partial_x [\mu_{\pm} F_{\pm}(x) C_{\pm}(x)] = D_{\pm} \partial_x^2 C_{\pm}(x)$$

Integrating one time gives

$$\begin{aligned} \partial_x C_{\pm}(x) &= \frac{\mu_{\pm}}{D_{\pm}} F_{\pm}(x) C_{\pm}(x) + \chi_1 \\ &= \mp \frac{e}{k_B T} \frac{d\phi(x)}{dx} C_{\pm}(x) + \chi_1 \end{aligned}$$

$\chi_1 = 0$  since  $\partial_x C_{\pm}(x)|_{x=\infty} = 0$  and  $\partial_x \phi(x)|_{x=\infty} = 0$ .

so that the eq. is separable:

$$\frac{\partial_x C_{\pm}(x)}{C_{\pm}(x)} = \mp \frac{e}{k_B T} \frac{d\phi}{dx}$$



Integrating once more gives:

$$\ln C_{\pm}(x) = \mp \frac{e}{k_B T} \phi(x) + \mathcal{A}_2$$

$$C_{\pm}(x) = \mathcal{A} \exp\left(\mp \frac{e}{k_B T} \phi(x)\right), \quad \mathcal{A} = e^{\mathcal{A}_2}$$

Boundary conditions  $C_{\pm}(x=\infty) = C_0, \phi(x=\infty) = 0 \Rightarrow \mathcal{A} = C_0$

$$C_{\pm}(x) = C_0 \exp\left(\mp \frac{e}{k_B T} \phi(x)\right)$$

c) The electrical potential:

$$-\epsilon \frac{d^2 \phi(x)}{dx^2} = e [C_+(x) - C_-(x)]$$

$$\epsilon \partial_x^2 \phi(x) = -e C_0 [e^{-e\phi/k_B T} - e^{e\phi/k_B T}]$$

$$\partial_x^2 \phi(x) = \frac{2eC_0}{\epsilon} \sinh\left(\frac{e\phi(x)}{k_B T}\right)$$

d) Linearize the eq. for the electrical potential:

$$\partial_x^2 \phi(x) = \frac{2eC_0}{\epsilon} \left(\frac{e\phi}{k_B T} + \dots\right) \approx \frac{2e^2 C_0}{\epsilon k_B T} \phi(x)$$

$$\partial_x^2 \phi(x) - k^2 \phi(x) = 0 \quad k^2 = \frac{2e^2 C_0}{\epsilon k_B T}$$

Solution

$$\phi(x) = A e^{kx} + B e^{-kx}$$

When  $x \rightarrow \infty$ ,  $\phi(x) \rightarrow 0$  which implies that  $A=0$

The boundary condition at  $x=0$  gives  $B = -\phi_0$ .

Hence

$$\phi(x) = -\phi_0 e^{-kx} = \underline{-\phi_0 e^{-x/\lambda}}$$

where

$$\lambda = k^{-1} = \sqrt{\frac{\epsilon k_B T}{2e^2 c_0}}$$

e] Charge density :

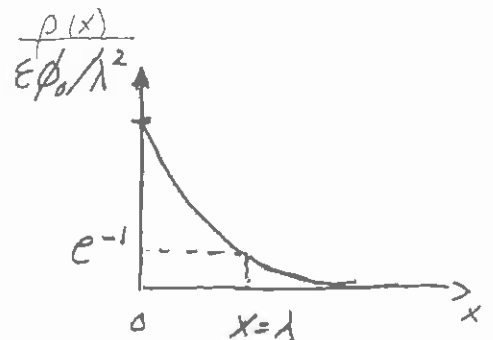
$$\rho(x) = e [c_+(x) - c_-(x)] = -2ec_0 \sinh\left(\frac{e\phi(x)}{k_B T}\right)$$

Linearized charge density :

$$\rho(x) \approx -2ec_0 \frac{e\phi(x)}{k_B T}$$

$$= \frac{2e^2 c_0}{k_B T} \phi_0 e^{-x/\lambda}$$

$$= \underline{\frac{\epsilon \phi_0}{\lambda^2} e^{-x/\lambda}}$$



$\lambda$  sets the scale over which  $\rho(x)$  has dropped to  $e^{-1}$  of its value at the wall. Since  $\rho(x) \rightarrow 0$ , far away from the wall, a screening effect is present. It is thus natural to call  $\lambda$  a screening length.