TFY 4275 / FY 8907

Classical Transport Theory

Final exam Wednesday, December 14, 2016

Solution Set

Problem 1

a) $$\langle A \rangle = A_0 Q_0 + \sum_{s=1}^{\infty} \frac{1}{s!} \int_{-\infty}^{+\infty} d\tau_1 \cdots d\tau_s \, A_s(\tau_1, \ldots, \tau_s) \, Q_s(\tau_1, \ldots, \tau_s)$$

b) $\sum_{\sigma=1}^{s} \chi(\tau_\sigma)$ counts the number of collisions that happens in the time interval $(0, t)$ out of a total of s collisions.

c) $$Q_0 + \sum_{s=1}^{\infty} \frac{1}{s!} \int_{-\infty}^{+\infty} d\tau_1 \cdots d\tau_s \, Q_s(\tau_1, \ldots, \tau_s)$$

$$= e^{-\nu} \sum_{s=0}^{\infty} \frac{1}{s!} \left(\int_{-\infty}^{+\infty} d\tau \, q(\tau) \right)^s$$

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$$= e^{-\nu + \int_{-\infty}^{+\infty} d\tau\, q(\tau)} = 1 \quad \Rightarrow$$

$$\underline{\underline{\int_{-\infty}^{+\infty} d\tau\, q(\tau) = \nu}}$$

d) ρ is the average number of collisions per time unit.

e) $$\underline{\underline{\langle E \rangle}} = E_0 \left\{ e^{-\nu} + e^{-\nu} \sum_{s=1}^{\infty} \frac{1}{s!} \left(\int_{-\infty}^{+\infty} d\tau\, q(\tau)\, \eta^{2\chi(\tau)} \right)^s \right\}$$

$$= E_0 e^{-\nu} \sum_{s=0}^{\infty} \frac{1}{s!} \left(\rho \int_{-\infty}^{+\infty} d\tau\, \eta^{2\chi(\tau)} \right)^s$$

$$= E_0\, e^{\rho \int_{-\infty}^{+\infty} d\tau\, \eta^{2\chi(\tau)} - \nu}$$

$$= E_0\, e^{\rho \int_{-\infty}^{+\infty} d\tau\, (\eta^{2\chi(\tau)} - 1)}$$

$$\underline{\underline{= E_0 e^{\rho(\eta^2 - 1)t}}}$$

f) $$\underline{\underline{\langle \nu \rangle}} = \nu_0\, e^{-\nu} \sum_{s=0}^{\infty} \frac{1}{s!} \left(\int_{-\infty}^{+\infty} d\tau\, \eta^{\chi(\tau)} \right)^s$$

$$= \nu_0\, e^{\rho \int_{-\infty}^{+\infty} d\tau\, \eta^{\chi(\tau)} - \nu}$$

$$\nu_s(\tau_1, \ldots, \tau_s) = \nu_0\, \eta^{\sum_{k=1}^{s} \chi(\tau_k)}$$

$$= \nu_0\, e^{\rho \int_{-\infty}^{+\infty} d\tau\, (\eta^{\chi(\tau)} - 1)} \underline{\underline{= \nu_0 e^{\rho(\eta - 1)t}}}$$

g) v_{RMS} is based on the second moment of v, whereas $\langle v \rangle$ is based on the first moment.

"The square of the average is not equal to the average of the square."

h) 1. The number of collisions per time must depend on the velocity; the slower the steel ball moves, the fewer the collisions per time. In our calculation, we have set ρ = constant.

2. The sand grains have a typical size and they form a random close packing. It is therefore an oversimplification to model the process as shot noise.

Problem 2

a) $P_{1|1}(y_3, t_3 \mid 1, t_2)\, P_{1|1}(1, t_2 \mid y_1, t_1)$

$+\, P_{1|1}(y_3, t_3 \mid -1, t_2)\, P_{1|1}(-1, t_2 \mid y_1, t_1)$

$$= \left((+)_{32}\, \delta_{y_3,1} + (-)_{32}\, \delta_{y_3,-1} \right) \left((+)_{21}\, \delta_{1,y_1} + (-)_{21}\, \delta_{1,-y_1} \right)$$

$$\boxed{\frac{1}{2}\left(1 \pm e^{-2\gamma(t_i - t_j)}\right) = (\pm)_{ij}}$$ 4

$$+ \left((+)_{32}\, \delta_{y_3,-1} + (-)_{32}\, \delta_{y_3,1} \right)$$

$$\left((+)_{21}\, \delta_{-1,y_1} + (-)_{21}\, \delta_{-1,-y_1} \right)$$

$$= (+)_{32}(+)_{21}\, \delta_{y_3,1}\, \delta_{1,y_1} + (+)_{32}(-)_{21}\, \delta_{y_3,1}\, \delta_{1,-y_1}$$

$$+ (-)_{32}(+)_{21}\, \delta_{y_3,-1}\, \delta_{1,y_1} + (-)_{32}(-)_{21}\, \delta_{y_3,-1}\, \delta_{1,-y_1}$$

$$+ (+)_{32}(+)_{21}\, \delta_{y_3,-1}\, \delta_{-1,y_1} + (+)_{32}(-)_{21}\, \delta_{y_3,-1}\, \delta_{-1,-y_1}$$

$$+ (-)_{32}(+)_{21}\, \delta_{y_3,+1}\, \delta_{-1,y_1} + (-)_{32}(-)_{21}\, \delta_{y_3,+1}\, \delta_{-1,-}$$

$$= \left((+)_{32}(+)_{21} + (-)_{32}(-)_{21} \right) \delta_{y_3,1}\, \delta_{y_1,1}$$

$$+ \left((-)_{32}(-)_{21} + (+)_{32}(+)_{21} \right) \delta_{y_3,-1}\, \delta_{y_1,-1}$$

$$+ \left((+)_{32}(-)_{21} + (-)_{32}(+)_{21} \right) \delta_{y_3,1}\, \delta_{y_1,-1}$$

$$+ \left((+)_{32}(-)_{21} + (-)_{32}(+)_{21} \right) \delta_{y_3,-1}\, \delta_{y_1,1}$$

$$= \left((+)_{32}(+)_{21} + (-)_{32}(-)_{21} \right) \left(\delta_{y_3,1}\, \delta_{y_1,1} + \delta_{y_3,-1}\, \delta_{y_1,-1} \right.$$
$$+ \left((+)_{32}(-)_{21} + (-)_{32}(+)_{21} \right) \left(\delta_{y_3,+1}\, \delta_{y_1,-1} + \delta_{y_3,-1}\, \delta_{y_1,1} \right.$$

$$= 2\left((+)_{32}(+)_{21} + (-)_{32}(-)_{21} \right) \delta_{y_3 y_1}$$
$$+ 2\left((+)_{32}(-)_{21} + (-)_{32}(+)_{21} \right) \delta_{y_3,-y_1}$$

$$= (+)_{31}\, \delta_{y_3 y_1} + (-)_{31}\, \delta_{y_3,-y_1} = \underline{P_{1|1}(y_3, t_3 | y_1, t_1)}$$

b) We use

$$P_1(y_2, t_2) = \int P_{1|1}(y_2, t_2 | y_1, t_1)\, P_1(y_1, t_1)\, dy_1$$

which here becomes

$$\underline{\underline{P_1(y_2, t_2)}} = P_{1|1}(y_2, t_2 | 1, t_1)\,\frac{1}{2}$$

$$+ P_{1|1}(y_2, t_2 | -1, t_1)\,\frac{1}{2}$$

$$= \frac{1}{2}(+)_{21}\,\delta_{y_2,1} + \frac{1}{2}(-)_{21}\,\delta_{y_2,-1}$$

$$+ \frac{1}{2}(+)_{21}\,\delta_{y_2,-1} + \frac{1}{2}(-)_{21}\,\delta_{y_2,1}$$

$$= \frac{1}{2}\left((+)_{21} + (-)_{21}\right)\delta_{y_2,1}$$

$$+ \frac{1}{2}\left((+)_{21} + (-)_{21}\right)\delta_{y_2,-1}$$

$$= \cancel{\tfrac{1}{2}(2)\cdot} = \underline{\underline{\frac{1}{2}\,\delta_{y_2,1} + \frac{1}{2}\,\delta_{y_2,-1}}}$$

c) The master equation is a <u>continuity equation</u> for the probability $P_1(y,t)$. The change in time of $P(y,t)$ equals the <u>flow into</u> $P(y,t)$ from the other states minus the <u>flow out of</u> $P(y,t)$ to the other states.

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d) $P_{1|1}(y_2, t_2 | y_1, t_1)$

$$= \frac{1}{2}\left(1 + e^{-2\gamma(t_2 - t_1)}\right)\delta_{y_2, y_1}$$

$$+ \frac{1}{2}\left(1 - e^{-2\gamma(t_2 - t_1)}\right)\delta_{y_2, -y_1}$$

$$= \frac{1}{2}\left(2 - 2\gamma(t_2 - t_1) + \cdots\right)\delta_{y_2, y_1}$$

$$+ \frac{1}{2}\left(0 + 2\gamma(t_2 - t_1) + \cdots\right)\delta_{y_2, -y_1}$$

$$= \left(1 - \gamma(t_2 - t_1)\right)\delta_{y_2, y_1} + \gamma(t_2 - t_1)\delta_{y_2, -y_1} + O(t_2 - t_1)^2.$$

$$\Rightarrow \quad \underline{W(y_2 | y_1) = \gamma\,\delta_{y_2, -y_1}}$$

Master equation:

$$\underline{\underline{\frac{\partial}{\partial t} P_1(y, t)}} = \sum_{y' = \pm 1}' \left[\gamma\,\delta_{y, -y'} P(y', t) - \gamma\,\delta_{y', -y} P(y, t)\right]$$

$$\underline{\underline{= \gamma\left(P(-y, t) - P(y, t)\right)}}$$

e) $\underline{\underline{\dot{P}_1(y)}} = \gamma \left(P(-y,t) - P(y,t) \right)$

$= \frac{1}{2} \gamma \left(\delta_{-y,1} + \delta_{-y,-1} - \delta_{y,1} - \delta_{y,-1} \right)$

$= \frac{1}{2} \gamma \left(\delta_{y,-1} + \delta_{y,1} - \delta_{y,1} - \delta_{y,-1} \right) \underline{\underline{= 0}}$