Exam TFY4275/FY8907
November 30, 2018
Solutions

Problem 1

a) $$H = \int d^3r \int d^3v \; f \ln f$$

$$\frac{dH}{dt} = \int d^3r \int d^3v \; (1+\ln f) \frac{\partial f}{\partial t}$$

$$= \int d^3r \int d^3v \; (1+\ln f)\left(\frac{\partial f}{\partial t}\right)_{coll}$$

$$- \underbrace{\int d^3r \int d^3v \; \underbrace{(1+\ln f)\, \vec{v}\cdot\frac{\partial f}{\partial \vec{r}}}_{=\vec{v}\cdot\frac{\partial}{\partial \vec{r}}(f\ln f)}}_{=0}$$

$$+ \underbrace{\int d^3r \int d^3v \; \frac{1}{m}\frac{\partial U}{\partial \vec{r}} \frac{\partial}{\partial \vec{v}}(f\ln f)}_{=0}$$

$$= \int d^3r \int d^3v \; (1+\ln f)\left(\frac{\partial f}{\partial t}\right)_{coll}$$

$$\frac{dH}{dt} = \int d^3r \int d^3v \int d^3v_1 \int d\Omega \; \sigma(\Omega)\, |\vec{v}_1 - \vec{v}| \, (f' f_1' - f f_1)(1+\ln f)$$

$\vec{v}_1 \rightleftarrows \vec{v}$:

$$\frac{dH}{dt} = \int d^3r \int d^3v \int d^3v_1 \int d\Omega \; \sigma(\Omega) (f' f_1' - f f_1) |\vec{v}_1 - \vec{v}| (1+\ln f_1)$$

$$\Rightarrow \frac{dH}{dt} = \frac{1}{2}\int d^3r \int d^3v \int d^3v_1 \int d\Omega \; \sigma(\Omega) (f' f_1' - f f_1)(2+\ln(f_1 f))\, |\vec{v}_1 - \vec{v}|$$

Next, we exploit invariance of $\frac{dH}{dt}$ under $\vec{v} \rightleftarrows \vec{v}'$; $\vec{v}_1 \rightleftarrows \vec{v}_1'$

$d^3v_1' \, d^3v' = d^3v_1 \, d^3v$

$\Rightarrow$

$$\frac{dH}{dt} = -\frac{1}{2} \int d^3r \int d^3v \int d^3v_1 \int d\Omega \, \sigma(\Omega) \, (f' f_1' - f f_1) \, |\vec{v}_1 - \vec{v}| \left[2 + \ln(f' f_1')\right]$$

Add the two versions of $\frac{dH}{dt}$:

$$\frac{dH}{dt} = \frac{1}{4} \int d^3r \int d^3v \int d^3v_1 \int d\Omega \, \sigma(\Omega) \, (f' f_1' - f f_1) \cdot |\vec{v}_1 - \vec{v}| \cdot \ln\left(\frac{f f_1}{f' f_1'}\right)$$

Now, $(x-y) \ln \frac{y}{x} \leq 0$

$(x-y) \ln \frac{y}{x} = 0 \Leftrightarrow y = x$

$$\boxed{\begin{aligned} &\frac{dH}{dt} \leq 0 \\ &\frac{dH}{dt} = 0 \Leftrightarrow f_1 f = f_1' f' \end{aligned}}$$

The physical interpretation of this result is that a system evolves irreversibly from an initial state towards some equilibrium state.

b) Start with the BE, multiply by A, and integrate over $\vec{v}$:

$$\int d^3v \left[A \left(\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} + \vec{a}\cdot\frac{\partial f}{\partial \vec{v}} \right) \right]$$

$$= \frac{\partial \langle n A\rangle}{\partial t} + \frac{\partial}{\partial \vec{r}}\cdot\langle n\vec{v}A\rangle + \underbrace{\int d^3v A\vec{a}\cdot\frac{\partial f}{\partial \vec{v}}}_{= -\vec{a}\cdot \underbrace{\int d^3v\, f\frac{\partial A}{\partial \vec{v}}}_{= -\vec{a}\cdot\langle n\frac{\partial A}{\partial \vec{v}}\rangle}}$$

$$= \int d^3v\, A\left(\frac{\partial f}{\partial t}\right)_{coll}$$

$$= \underline{\underline{\int d^3v \int d^3v_1 \int d\Omega\, \sigma(\Omega)\,|\vec{v}_1-\vec{v}|\,(f'f_1'-ff_1)\,A(\vec{v})}}$$

c) Collision-invariant:

$$\underline{\underline{A(\vec{v}_1)+A(\vec{v}) = A(\vec{v}_1')+A(\vec{v}')}}$$

Define the above integral as I_1

$v_1 \rightleftarrows \vec{v} \Rightarrow$

$$I_1 = \frac{1}{2}\int d^3v \int d^3v_1 \int d\Omega\, \sigma(\Omega)\,|\vec{v}_1-\vec{v}|\,(f'f_1'-ff_1)\,(A+A_1)$$

$\vec{v} \rightleftarrows \vec{v}'$; $\vec{v}_1 \rightleftarrows \vec{v}_1'$

$$I = \frac{1}{2}\int d^3v \int d^3v_1 \int d\Omega\, \sigma(\Omega)\,|\vec{v}_1-\vec{v}|\,(ff_1-f'f_1')\,(A'+A_1')$$

Now add these 2

$$I_1 = \frac{1}{4}\int d^3v \int d^3v_1 \int d\Omega\, \sigma(\Omega)\,|\vec{v}_1-\vec{v}|\,(f'f_1'-ff_1)\,(A+A_1-(A'+A_1'))$$

If A is a collision-inv. $\Rightarrow$

$A + A_1 = A_1' + A' \Rightarrow I_1$ vanishes

Then we have the conservation law

$$\frac{\partial \langle nA \rangle}{\partial t} + \frac{\partial}{\partial \vec{r}} \cdot \left(n \vec{v} A \right) - \vec{a} \cdot \left(n \frac{\partial A}{\partial \vec{v}} \right) = 0$$

d) $\vec{A} = \vec{p}$

$$\frac{\partial (n \langle p_i \rangle)}{\partial t} + \frac{\partial}{\partial x_j} \left(\frac{n}{m} \langle p_i p_j \rangle \right)$$

$$- \frac{n F_j}{m} \delta_{ij} m = 0$$

$$\frac{\partial (\rho \langle v_i \rangle)}{\partial t} + \frac{\partial}{\partial x_j} \left(\rho \langle v_i v_j \rangle \right) - n F_i = 0$$

This is Newton's second law for a small fluid-element, where F_i is an external force and the second term is the force acting on the fluid element from the surrounding fluid.

e)

$\langle v_i \rangle = u_i$

$\langle v_i v_j \rangle = u_i u_j + \langle (v_i - u_i)(v_j - u_j) \rangle$

Insert back into equation for $\vec{p}$:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho\, u_i u_j\right)$$

$$= n F_i - \frac{\partial}{\partial x_j}\left[\rho \langle (v_i - u_i)(v_j - u_j) \rangle\right]$$

$$\to \vec{F} - \vec{\nabla} \cdot \overleftrightarrow{P}$$

$$\underline{\underline{P_{ij} = \rho \langle (v_i - u_i)(v_j - u_j) \rangle}}$$

f) Define $U_i = v_i - u_i$
(fluctuations around average)

$$f(\vec{r}, \vec{v}, t) = n \left(\frac{m}{2 k_B \pi T}\right)^{3/2} e^{-\frac{m}{2k_BT} U^2}$$

$$P_{ij} = \frac{\rho}{n}\, n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int d^3U\, U_i U_j\, e^{-\frac{m}{2k_BT} U^2}$$

$$= \delta_{ij}\, \rho \left(\frac{m}{2\pi k_B T}\right) \frac{1}{3} \int d^3U\, U^2 e^{-\alpha U^2}$$

$$= \delta_{ij}\, \rho \left(\frac{m}{2\pi k_B T}\right)^{3/2} \frac{4\pi}{3} \underbrace{\int_0^\infty dU\, U^4 e^{-\alpha U^2}}_{= \frac{3}{8} \frac{1}{\alpha^2} \sqrt{\frac{\pi}{\alpha}}}$$

$\alpha = \frac{m}{2k_BT}$

$$P_{ij} = \rho \quad \delta_{ij} \left(\frac{\alpha}{\pi}\right)^{3/2} \frac{3}{8}\sqrt{\frac{\pi}{\alpha}} \frac{1}{\alpha^2} \frac{4\pi}{3}$$

$$= \delta_{ij} \rho \frac{1}{2\alpha} = \delta_{ij} \rho \frac{k_B T}{m}$$

$$= \delta_{ij} \; n \, m \frac{k_B T}{m} = \delta_{ij} \, n k_B T$$

$$= \underline{\underline{\delta_{ij} \; p}} \; ; \quad p = n k_B T$$

p: Hydrostatic pressure, ideal gas

From this, it follows that

$$\underline{\underline{\eta = \zeta = 0}}$$

Problem 2

a) Detailed balance

$$\underline{w_{nn'} P_{n'}^0 = w_{n'n} P_n^0}$$

$(\phi, W\psi) =$

$$\sum_n \frac{1}{P_n^0} \phi(n) \sum_{n'} \left(w_{nn'} \psi(n') - w_{n'n} \psi(n) \right)$$

$$= \sum_n \sum_{n'} \frac{\phi(n)}{P_n^0} \left(w_{nn'} \psi(n') - w_{n'n} \psi(n) \right)$$

$(\psi, M\phi)$

$$= \sum_{n,n'} \frac{\psi(n)}{P_n^0} \left(w_{nn'} \phi(n') - w_{n'n} \phi(n) \right)$$

Now use detailed balance in the first term after interchanging n, n'

$$= \sum_{n,n'} \frac{\phi(n)}{P_n^0} \left(w_{nn'} \psi(n') - w_{n'n} \psi(n) \right)$$

$$= \underline{(\phi, W\psi)}$$

$$P_n(t) = \sum_\lambda c_\lambda(t) \, \phi_\lambda$$

$$\dot{P}_n = W P_n$$

$$I = (\Phi_\lambda, W\Phi_\lambda) = -\lambda$$

$$I = \sum_{n,n'} \frac{\Phi_\lambda(n)}{P_n^0} \left(\omega_{nn'} \Phi_\lambda(n') - \omega_{n'n} \Phi_\lambda(n) \right)$$

Introduce $\quad g(n) = \dfrac{\Phi_\lambda(n)}{P_n^0}$

$$I = \sum_{n,n'} g(n) \left(\omega_{nn'} P_{n'}^0 \, g(n') - \omega_{n'n} P_n^0 \, g(n) \right)$$

use detailed balance in first term

$$I = \sum_{n,n'} g(n^*) \, \omega_{n'n} P_n^0 \left(g(n') - g(n) \right)$$

I is invariant under interchange $n \rightleftarrows n'$

Using detailed balance :

$$I = \sum_{n,n'} g(n') \, \omega_{n'n} P_n^0 \left(g(n) - g(n') \right)$$

Now add these $\Rightarrow$

$$I = -\frac{1}{2} \sum_{n,n'} \omega_{nn'} P_{n'}(\partial) \left| g(n') - g(n) \right|^2 \leq 0$$

$$= -\lambda \Rightarrow \underline{\underline{\lambda \geq 0}}$$

$\lambda = 0 \Rightarrow \quad g(y) = g(y') \quad$ for all y, y'

$$\Rightarrow \quad \underline{\underline{P_n^0 = \Phi_0(n)}}$$

$$\sum_{\lambda} \dot{c}_\lambda \, \phi_\lambda = -\sum_{\lambda} c_\lambda(t)\, \lambda\, \phi_\lambda$$

W is hermitian $\Longrightarrow$

$\phi_\lambda \perp \phi_{\lambda'}$; $\lambda \neq \lambda'$

$$\dot{c}_\lambda = -\lambda\, c_\lambda$$

$$c_\lambda(t) = \underbrace{c_\lambda(0)}_{c_\lambda} e^{-\lambda t} \quad ; \quad \lambda \geq 0$$

$$P_n(t) = \sum_{\lambda \geq 0} c_\lambda\, e^{-\lambda t}\, \phi_\lambda$$

$$= c_0\, \phi_0 + \sum_{\lambda > 0} c_\lambda\, e^{-\lambda t}\, \phi_\lambda$$

$$c_\lambda = \left(\phi_\lambda, P_n(0)\right)$$

$$c_0 = \left(\phi_0, P_0\right)$$

$$\phi_0 = P_0 \Longrightarrow$$

Assuming normalization $\Longrightarrow c_0 = 1$

$$P_n(t) = \phi_0 + \sum_{\lambda > 0} c_\lambda\, e^{-\lambda t}\, \phi_\lambda$$

$t \to \infty$

$$\underline{\underline{P_n(t) \to P_0 = \phi_0}}$$

b) $$w_{n,n'} = \alpha\, \delta_{n+1,n'} + \beta\, \delta_{n-1,n'}$$

α: Probability per unit time (transition rate) for $n \to n-1$ (recombination step)

β: Same for $n \to n+1$ (generation step)

Master-equation

$$\dot{P}_n = \alpha P_{n+1} + \beta P_{n-1} - (\alpha+\beta) P_n$$

Balance condition

$$\alpha P_{n+1} + \beta P_{n-1} = (\alpha+\beta) P_n$$

Detailed balance

$$\alpha P_{n+1} = \beta P_n$$
$$\alpha P_n = \beta P_{n-1}$$

These two are equivalent.

To find $P_n(t)$, we write $F(z,t)$ as a power-series in z:

Consider the z-dependent terms:

$$e^{\frac{\alpha}{z}t}\; e^{\beta z t}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha t)^k z^{-k} \sum_{l=0}^{\infty} \frac{1}{l!} (\beta t)^l z^l$$

$$= \sum_{\substack{k\geq 0 \\ l\geq 0}} \frac{1}{l!} \frac{1}{k!} (\alpha t)^k (\beta t)^l \; z^{l-k}$$

Define $n = l-k \Rightarrow l = n+k$

$k, l \geq 0$

$n \in \mathbb{Z}$

$$F(z,t) = e^{-(\alpha+\beta)t} \sum_n \left(\sum_k \frac{(\alpha t)^k (\beta t)^{n+k}}{k!(n+k)!} \right) z^n$$

$$P_n(t) = \sum_{k=0}^{\infty} \frac{(\alpha t)^k (\beta t)^{n+k}}{k!\,(n+k)!} \; e^{-(\alpha+\beta)t}$$

$\alpha = \beta = q$

$$P_n(t) = e^{-2qt} \sum_{k=0}^{\infty} \frac{(qt)^{n+2k}}{k!\,(n+k)!}$$

Symm. random walk

$$\left.\begin{array}{l}\alpha = 0 \\ \beta = q\end{array}\right\} \text{Poisson-process}$$

In the sum entering $P_n(t)$,
only the $k=0$-term contributes
when $\alpha = 0$

$$\underline{\underline{P_n(t) = e^{-qt} \frac{(qt)^n}{n!}}} \quad \text{Poisson-distribution}$$

d) $$\langle n \rangle = \sum_n n P_n$$

$$= z \left. \frac{\partial F}{\partial z} \right|_{z=1}$$

$$F(z,t) = e^{-(\alpha+\beta)t} \, e^{\frac{\alpha}{z} t + \beta z t}$$

$$z \frac{\partial F}{\partial z} = e^{-(\alpha+\beta)t} \cdot z \, e^{\left(\frac{\alpha}{z} t + \beta z t\right)} \left(\beta - \frac{\alpha}{z^2}\right) t$$

$z = 1$:

$$z \frac{\partial F}{\partial z} = (\beta - \alpha) t$$

$$\underline{\underline{\langle n \rangle = (\beta - \alpha) t}}$$

$$\underline{\underline{V_d = \beta - \alpha}}$$

$$\frac{\partial^2 F}{\partial z^2} = e^{-(\alpha+\beta)t} \, e^{\frac{\alpha}{z}t + \beta z t} \cdot \left[\left(\left(\beta - \frac{\alpha}{z^2} \right) t \right)^2 + \frac{2\alpha t}{z^3} \right]$$

$z = 1$:

$$\left. \frac{\partial^2 F}{\partial z^2} \right|_{z=1} = \langle n(n-1) \rangle$$

$$= (\beta - \alpha)^2 t^2 + 2\alpha t$$

$$\langle (n - \langle n \rangle)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2$$

$$= \langle n(n-1) \rangle + \langle n \rangle - \langle n \rangle^2$$

$$= (\beta - \alpha)^2 t^2 + 2\alpha t + (\beta - \alpha) t - (\beta - \alpha)^2 t^2$$

$$= (\beta + \alpha) t$$

$$\underline{\underline{D = \alpha + \beta}}$$