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TFY4280 Signalanalyse (Signal Processing)

Examination May 19th, 2009, Time: 09.00 – 13.00

Allowed aid
NTNU allowed standard mathematics/physics tables (Rottman/Barnett and Cronin)
Electronics: NTNU allowed calculator.

Contents:
Examination problems (4 pages with this page)
Part A: only answers should be handed in (1 page).
Part B: solutions and answers should be handed in (2 pages).
Attachment (6 pages).

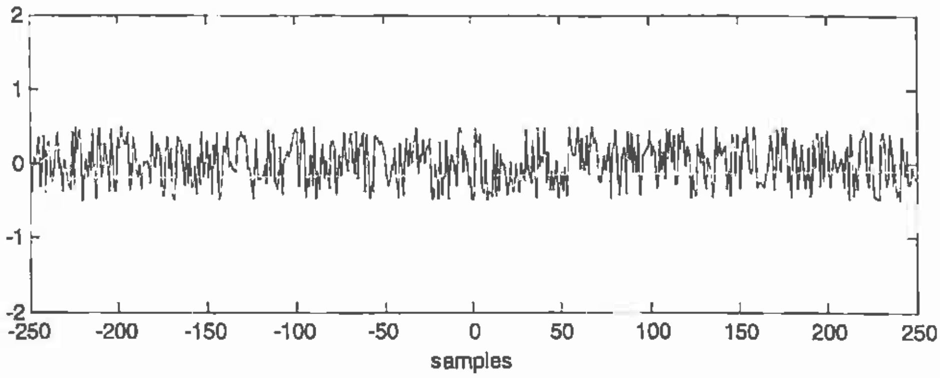
Grades to be announced before June 6th, 2009.

Evaluation/grades
Total number of points of the written examination is 100. The following table recommended by NTNU will be used for converting to A, B, C, ...-scale.
A: 100-90 points
B: 89-80 points
C: 79-60 points
D: 59-50 points
E: 49-40 points
F: 39-0 points

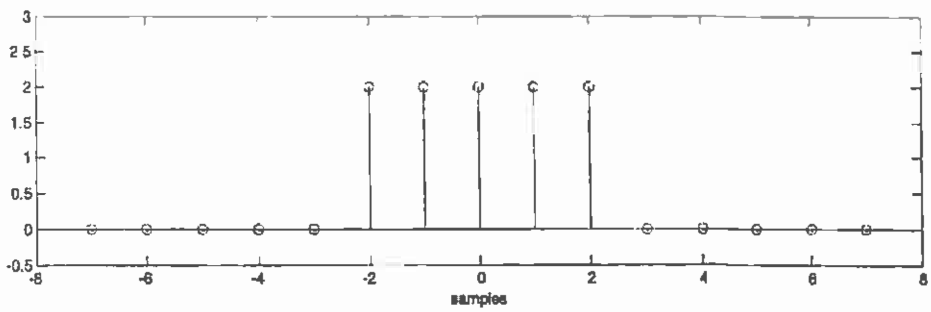
Section A: 4 problems to which only answers shall be given (each max 5 points)

A1: Sketch the self-convolution **AND** self-correlation of noise:

Typical noise signal:



A2: Sketch self-convolution of the following digitized 'rectangular' function:



Place the result centred on '0'.

A3: Draw the two-sided amplitude line spectrum and phase spectrum of the following signal:

$$3\cos\left(2\pi\cdot 30t+\frac{\pi}{3}\right)+2\cos\left(2\pi\cdot 80t+\frac{\pi}{6}\right)$$

A4: The Laplace transform of a signal is given by: $F(s)=\dfrac{2(s+3)}{s(s+1)(s+6)}$

Determine the signal waveform.

Section B: 4 problems to which complete solutions must be handed in (each max 20 points)

B1: A simple digital signal processing system is defined by: $x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$

The system impulse response is given by: $h[n]=\left\{\underset{\uparrow}{0.5},1.5,1\right\}$ (the arrow indicates 'zero')

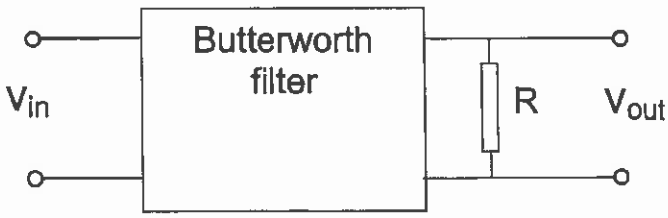
a) What is the response $y[n]$ to the unit step function: $u[n]=\left\{\underset{\uparrow}{1},1,1,1,1,.....\right\}$

b) What is the response to the input signal: $x[n]=\left\{2,\underset{\uparrow}{1},1.5,0.5\right\}$

B2: A second order Butterworth filter has a magnitude frequency response given by:

$$|H(\omega)|=\frac{1}{\sqrt{1+\left(\omega/\omega_c\right)^4}}=\frac{\omega_c^2}{\sqrt{\omega^4+\omega_c^4}} \qquad \text{eq. 1}$$

You are about to design a Butterworth filter using one inductance (L) and one capacitance (C) that passes very low (also zero) frequencies over the load resistance **R**, and has a cut-off frequency of 100 rad/s, see the scheme below:



a) Draw the appropriate circuit (inside the filter).

b) Derive the transfer function (eq.1) and show that for the Butterworth filter in question:

$$\frac{1}{LC}=\omega_c^2 \text{ and } L=2R^2C$$

c) Using a load resistance of 1000 Ω and the cut-off frequency 100 rad/s, calculate L and C.

d) Calculate the frequency response of the signal for ω = 0, 100 and 377 rad/s.

B3: a) Show that the time-transformation property of the Laplace transform (single sided) is:

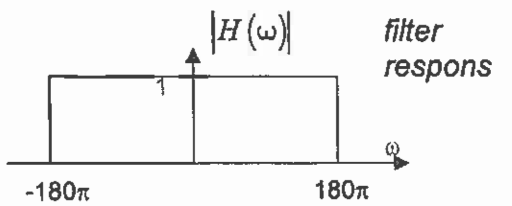
$$L\left[f(at-b)u(at-b)\right]=\frac{e^{-sb/a}}{a}F\left(\frac{s}{a}\right), \qquad a>0;\ b\geq 0$$

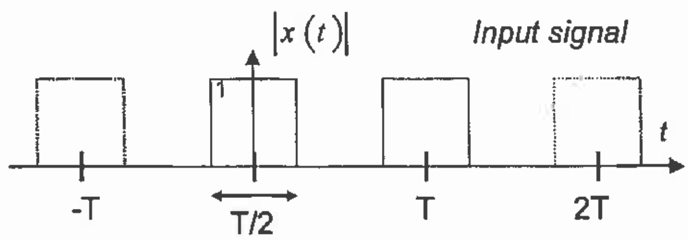
u(t) is the unit step function.

b) Find the function *f(t)* having the Laplace transform: $L\left[f(t)\right]=\frac{3}{s^2+9}$.

c) Show that the corresponding time-transformed function $f\left(4t-\frac{\pi}{6}\right)\cdot u\left(4t-\frac{\pi}{6}\right)$ can be written: $\sin\left[12\left(t-\frac{\pi}{24}\right)\right]\cdot u\left(t-\frac{\pi}{24}\right)$

B4: As illustrated below, a periodic square wave is the input signal to an ideal low-pass filter with the depicted frequency response. $x(t)\rightarrow\boxed{\begin{matrix}\text{ideal } LP\\ |H(\omega)|\end{matrix}}\rightarrow y(t)$





a) Find the Fourier transform of the input function.

b) What is the output signal if the input signal has the period T = 40 ms?

Hint - The FT of a periodic function can be written:

$$F\left[\sum_{n=-\infty}^{\infty} g(t-nT_0)\right]=\sum_{n=-\infty}^{\infty}\omega_0 G(n\omega_0)\delta(\omega-n\omega_0);\ \omega_0=\frac{2\pi}{T_0}$$

TABLE 5.1 Fourier Transform Properties

Operation	Time Function	Fourier Transform
Linearity	$af_1(t) + bf_2(t)$	$aF_1(\omega) + bF_2(\omega)$
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Time scaling	$f(at)$	$\frac{1}{\lvert a \rvert} F\left(\frac{\omega}{a}\right)$
Time transformation	$f(at - t_0)$	$\frac{1}{\lvert a \rvert} F\left(\frac{\omega}{a}\right) e^{-j\omega t_0/a}$
Duality	$F(t)$	$2\pi f(-\omega)$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega - \omega_0)$
Convolution	$f_1(t)*f_2(t)$	$F_1(\omega)F_2(\omega)$
	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$
Differentiation	$\frac{d^n[f(t)]}{dt^n}$	$(j\omega)^n F(\omega)$
	$(-jt)^n f(t)$	$\frac{d^n[F(\omega)]}{d\omega^n}$
Integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$

TABLE 5.2 Fourier Transform Pairs

Time Domain Signal	Fourier Transform
$f(t)$	$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$
$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$	$F(\omega)$
$\delta(t)$	1
$A\delta(t - t_0)$	$Ae^{-j\omega t_0}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
1	$2\pi\delta(\omega)$
K	$2\pi K\delta(\omega)$
$\mathrm{sgn}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$\mathrm{rect}(t/T)$	$T\mathrm{sinc}(\omega T/2)$
$\cos(\omega_0 t)u(t)$	$\frac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$
$\sin(\omega_0 t)u(t)$	$\frac{\pi}{2j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$
$\mathrm{rect}(t/T)\cos(\omega_0 t)$	$\frac{T}{2}\left[\mathrm{sinc}\left(\frac{(\omega - \omega_0)T}{2}\right) + \mathrm{sinc}\left(\frac{(\omega + \omega_0)T}{2}\right)\right]$
$\frac{\beta}{\pi}\mathrm{sinc}(\beta t)$	$\mathrm{rect}(\omega/2\beta)$
$\mathrm{tri}(t/T)$	$T\,\mathrm{sinc}^2(T\omega/2)$
$\mathrm{sinc}^2(Tt/2)$	$\frac{2\pi}{T}\mathrm{tri}(\omega/T)$
$e^{-at}u(t), \mathrm{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$
$te^{-at}u(t), \mathrm{Re}\{a\} > 0$	$\left(\frac{1}{a + j\omega}\right)^2$
$t^{n-1}e^{-at}u(t), \mathrm{Re}\{a\} > 0$	$\frac{(n-1)!}{(a + j\omega)^n}$
$e^{-a\lvert t\rvert}, \mathrm{Re}\{a\} > 0$	$\frac{2a}{a^2 + \omega^2}$
$\sum_{n=-\infty}^{\infty} g(t - nT_0)$	$\sum_{n=-\infty}^{\infty} \omega_0 G(n\omega_0)\delta(\omega - n\omega_0), \omega_0 = \frac{2\pi}{T_0}$
$\sum_{n=-\infty}^{\infty} g(t - nT_0) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$	$H(\omega) = \lvert H(\omega)\rvert\angle\phi(\omega) = \frac{\lvert V_2(\omega)\rvert\angle V_2}{\lvert V_1(\omega)\rvert\angle V_1} = \frac{\lvert V_2(\omega)\rvert}{\lvert V_1(\omega)\rvert}\angle V_2 - \angle V_1,$
$\delta_T(t)$	$\sum_{k=-\infty}^{\infty} \omega_0\delta(\omega - k\omega_0)$

TABLE 7.2 Laplace Transforms

$f(t),\ t \geq 0$	$F(s)$	ROC
1. $\delta(t)$	1	All s
2. $u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
3. t	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
4. t^n	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
5. e^{-at}	$\frac{1}{s+a}$	$\text{Re}(s) > -a$
6. te^{-at}	$\frac{1}{(s+a)^2}$	$\text{Re}(s) > -a$
7. $t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}(s) > -a$
8. $\sin bt$	$\frac{b}{s^2+b^2}$	$\text{Re}(s) > 0$
9. $\cos bt$	$\frac{s}{s^2+b^2}$	$\text{Re}(s) > 0$
10. $e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$	$\text{Re}(s) > -a$
11. $e^{-at}\cos bt$	$\frac{s+a}{(s+a)^2+b^2}$	$\text{Re}(s) > -a$
12. $t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$	$\text{Re}(s) > 0$
13. $t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$	$\text{Re}(s) > 0$

TABLE 7.3 Laplace Transform Properties

Name	Property
1. Linearity.	$\mathcal{L}[a_1f_1(t) + a_2f_2(t)] = a_1F_1(s) + a_2F_2(s)$
2. Derivative.	$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+)$
3. *n*th-order derivative.	$\mathcal{L}\left[\frac{d^nf(t)}{dt^n}\right] = s^nF(s) - s^{n-1}f(0^+)$ $- \cdots - sf^{(n-2)}(0^+) - f^{(n-1)}(0^+)$
4. Integral.	$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$
5. Real shifting.	$\mathcal{L}[f(t - t_0)u(t - t_0)] = e^{-t_0s}F(s)$
6. Complex shifting.	$\mathcal{L}[e^{-at}f(t)] = F(s + a)$
7. Initial value.	$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$
8. Final value.	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$
9. Multiplication by *t*.	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
10. Time transformation. $(a > 0; b \geqq 0)$	$\mathcal{L}[f(at - b)u(at - b)] = \frac{e^{-sb/a}}{a}F\left(\frac{s}{a}\right)$
11. Convolution	$\mathcal{L}^{-1}[F_1(s)F_2(s)] = \int_0^t f_1(t - \tau)f_2(\tau)\,d\tau$ $= \int_0^t f_1(\tau)f_2(t - \tau)d\tau$
12. Time periodicity $[f(t) = f(t + T)], t \geqq 0$	$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-sT}}F_1(s)$, where $F_1(s) = \int_0^T f(t)e^{-st}\,dt$

z-Transforms

$f[n], n \geqq 0$	$F(z)$	ROC
1. $\delta[n]$	1	All z
2. $\delta[n - n_0]$	z^{-n_0}	$z \neq 0$
3. $u[n]$	$\dfrac{z}{z - 1}$	$\lvert z \rvert > 1$
4. n	$\dfrac{z}{(z - 1)^2}$	$\lvert z \rvert > 1$
5. n^2	$\dfrac{z(z + 1)}{(z - 1)^3}$	$\lvert z \rvert > 1$
6. a^n	$\dfrac{z}{z - a}$	$\lvert z \rvert > \lvert a \rvert$
7. na^n	$\dfrac{az}{(z - a)^2}$	$\lvert z \rvert > \lvert a \rvert$
8. n^2a^n	$\dfrac{az(z + a)}{(z - a)^3}$	$\lvert z \rvert > \lvert a \rvert$
9. $\sin bn$	$\dfrac{z \sin b}{z^2 - 2z \cos b + 1}$	$\lvert z \rvert > 1$
10. $\cos bn$	$\dfrac{z(z - \cos b)}{z^2 - 2z \cos b + 1}$	$\lvert z \rvert > 1$
11. $a^n \sin bn$	$\dfrac{az \sin b}{z^2 - 2az \cos b + a^2}$	$\lvert z \rvert > \lvert a \rvert$
12. $a^n \cos bn$	$\dfrac{z(z - a \cos b)}{z^2 - 2az \cos b + a^2}$	$\lvert z \rvert > \lvert a \rvert$

Properties of the z-Transform

Name	Property
1. Linearity, (11.8)	$\mathscr{Z}[a_1 f_1[n] + a_2 f_2[n]] = a_1 F_1(z) + a_2 F_2(z)$
2. Real shifting, (11.13)	$\mathscr{Z}[f[n - n_0]u[n - n_0]] = z^{-n_0} F(z), \quad n_0 \geqq 0$
3. Real shifting, (11.25)	$\mathscr{Z}[f[n + n_0]u[n]] = z^{n_0}[F(z) - \sum_{n=0}^{n_0-1} f[n]z^{-n}]$
4. Complex shifting, (11.23)	$\mathscr{Z}[a^n f[n]] = F(z/a)$
5. Multiplication by n	$\mathscr{Z}[nf[n]] = -z \dfrac{dF(z)}{dz}$
6. Time scaling, (11.33)	$\mathscr{Z}[f[n/k]] = F(z^k)$, k a positive integer
7. Convolution, (11.38)	$\mathscr{Z}[x[n] * y[n]] = X(z)Y(z)$
8. Summation	$\mathscr{Z}[\sum_{k=0}^{n} f[k]] = \dfrac{z}{z - 1} F(z)$
9. Initial value, (11.27)	$f[0] = \lim_{z \to \infty} F(z)$
10. Final value, (11.30)	$f[\infty] = \lim_{z \to 1} (z - 1)F(z)$, if $f[\infty]$ exists