Norwegian University of Science and Technology
Department of Physics

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EXAM
TFY4280 Signal Processing

Sat. 2 June 2012. 09:00

Examination support materials:

- Simple calculator (according to NTNU exam regulations)
- K. Rottmann: Matematisk formelsamling (eller tilsvarende)
- Carl Angell og Bjørn Ebbe Lian: Fysiske størrelser og enheter, navn og symboler (eller tilsvarende)

Answer must be written in English or Norwegian. Number of points given to each sub-question is given in bold font. The maximum score for the exam is **100p**. The exam consists of 4 questions. Attachment: 2 pages with transform tables and properties.

Q1 (25p)

A) (15p) Calculate response (output $y(t)$) for a unit step function input ($\varepsilon(t)$) and delta impulse input $\delta(t)$ from a given impulse response function h_1 in the time domain:

$$h_1(t) = \left(e^{-t} - e^{-2t}\right)\varepsilon(t)$$

B) (10p) How would you describe the output ($y(t)$) of this LTI system, when a random signal $x(t)$ described by its μ_x and $\varphi_{xx}(\tau)$ is the input signal. Calculate μ_y and explain how to calculate $\varphi_{yy}(\tau)$ from known $h_1(t)$.

Q2 (25p) Consider LTI system described by:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 5\frac{\mathrm{d}}{\mathrm{d}t} + 4\right] y(t) = \left[2\frac{\mathrm{d}}{\mathrm{d}t} + 6\right] x(t)$$

A) (10p) Find the impulse response $h(t)$

B) (10p) Find the unit step response $s(t)$ by using $\epsilon(t)$ as input

C) (5p) Verify your result by showing that $h(t) = \frac{\mathrm{d}}{\mathrm{d}t}s(t)$

Q3 (25p) Explain the difference between FT, DTFT and DFT with respect to time domain signals for which those are calculated and resulting frequency representations. In a frequency range $\omega = \frac{2\pi}{f} \in [-30, 30]$, sketch absolute values of FT, DTFT and DFT of a signal defined by:

$$y(t) = e^{-a^2t^2}$$

for $a = 2\text{s}^{-1}$ and, where necessary, using sampling time $t_s = 0.25\text{s}$ and signal duration $t \in [-3, 3]$.

HINT:

$$F_s(\omega) = 2\pi\omega_s \sum_{n=-\infty}^{\infty} F(\omega - n\omega_s)$$

Q4 (25p) The figure below (Figure 4) shows a digital filter (DSP) in which the delays are 0.5 ms.

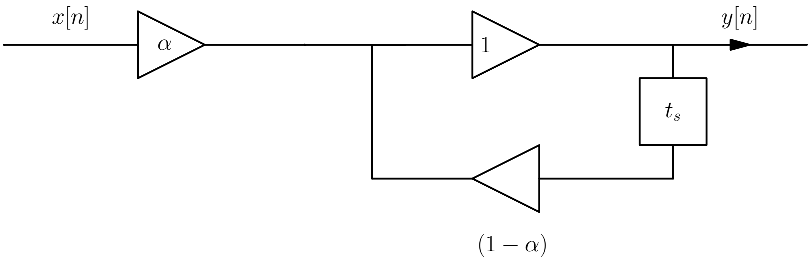

Figure 1: Question Q4

A) (10p) Write down the difference equation and from this derive Z-transform of the transfer function. Using a method of choice, analyse the system and calculate 5 first output terms $(0 \leq n < 5)$ for the unit step excitation and $\alpha = 0.1535$.

B) (15p) Now you would like to design an analogue 1st order Butterworth filter (using one capacitance C and one resistance R $= 1000\Omega$) with approximately the same frequency response. Determine needed capacitance C and plot filter circuit diagram.

HINT Derive expression for inpulse response of the DSP and Butterworth filters. For filters with similar frequency response, the impulse response functions will depend on time in the same manner $(h(t)/h(0) = h[n]/h[0])$. This also might be useful:

$$\begin{aligned} a^n &= \left(e^{\beta}\right)^n \quad \ln a = \beta \\ t &= n \cdot t_s \end{aligned}$$

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Examination support materials:

- Enkel kalkulator i henhold til NTNU eksamens regler
- K. Rottmann: Matematisk formelsamling (eller tilsvarende)
- Carl Angell og Bjørn Ebbe Lian: Fysiske størrelser og enheter, navn og symboler (eller tilsvarende)

Besvarelsen leveres på norsk eller engelsk. Antall poeng for hvert delspørsmål er gitt i uthevet font. Max antall poeng for hele eksamen er 100. Eksamen består av 4 oppgaver. Vedlegg: 2 sider med tabeller og egenskaper for transformasjonsfunksjoner.

Q1 (25p)

A) (15p) Beregn respons (utgangssignal y(t)) til en "unit step" funksjon ($x(t) = \varepsilon(t)$) og delta impuls $x(t) = \delta(t)$ fra en gitt impuls responsfunksjon h_1 i tidsdomenet:

$$h_1(t) = \left(e^{-t} - e^{-2t}\right) \varepsilon(t)$$

B) (10p) Hvordan vil du beskrive utgangssignal ($y(t)$) til dette LTI systemet, når et tilfeldig signal $x(t)$ beskrevet av μ_x og $\varphi_{xx}(\tau)$ er inngangssignal. Beregn μ_y og forklar hvordan man kan beregne $\varphi_{yy}(\tau)$ fra $h_1(t)$.

Q2 (25p) Et LTI system er beskrives med:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 5\frac{\mathrm{d}}{dt} + 4\right] y(t) = \left[2\frac{\mathrm{d}}{\mathrm{d}t} + 6\right] x(t)$$

A) (10p) Finn impulsrespons $h(t)$

B) (10p) Finn "unit step" respons $s(t)$ ved hjelp av $\epsilon(t)$ som input

C) (5p) Kontroller resultatet ved å vise at $h(t) = \frac{\mathrm{d}}{\mathrm{d}t}s(t)$

Q3 (25p) Forklar forskjellen mellom FT, DTFT og DFT med hensyn til tidsdomene signaler og resulterende frekvens representasjoner. I et frekvensområde $\omega = \frac{2\pi}{f} \in [-30, 30]$, skisse absolutte verdier av FT, DTFT og DFT av signalet $y(t)$:

$$y(t) = e^{-a^2t^2}$$

$a = 2\text{s}^{-1}$ og om nødvendig, ved hjelp av "sampling time" $t_s = 0.25\text{s}$ og signal varighet $t \in [-3, 3]$.
HINT:

$$F_s(\omega) = 2\pi\omega_s \sum_{n=-\infty}^{\infty} F(\omega - n\omega_s)$$

Q4 (25p) Figuren nedenfor (figur 2) viser et digitalt filter (DSP) hvor t_s er 0,5 ms.

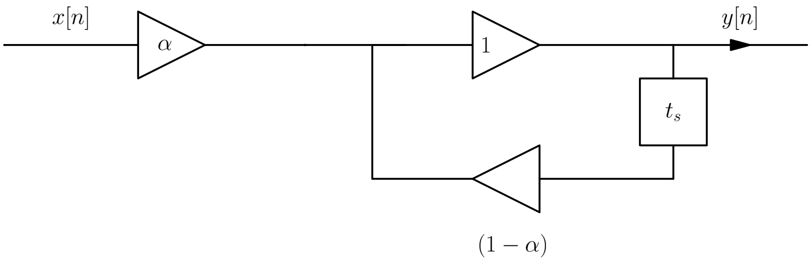

Figure 2: Spørsmål Q4

A) (10p) Skriv ned differanse ligningen og fra dette utled Z-transform av overføring funksjonen. Ved hjelp av utvalg metode, analyser systemet og beregn de fem første utgangs verdier $(0 \leq n < 5)$ for "unit step excitation" og $\alpha = 0.1535$.

B) (15p) Nå ønsker du å designe en analog "1st ordre" Butterworth filter (med kapasitans C og en motstand R = 1000Ω) med omtrent samme frekvens respons. Bestem nødvendig kapasitans C og plott filter koblingsskjema.

HINT Utled uttrykk for impulsrespons av DSP og Butterworth filter. For filtre med lignende frekvensrespons, vil impulsrespons funksjoner avhenger av tid på samme måte $(h(t)/h(0) = h[n]/h[0])$. Dette også kan være nyttig:

$$\begin{aligned} a^n &= \left(e^{\beta}\right)^n \quad \ln a = \beta \\ t &= n \cdot t_s \end{aligned}$$

Appendix B.1 Bilateral Laplace Transform Pairs

$x(t)$	$X(s)=\mathcal{L}\{x(t)\}$	ROC
$\delta(t)$	1	$s\in\mathbb{C}$
$\varepsilon(t)$	$\frac{1}{s}$	$\text{Re}\{s\}>0$
$e^{-at}\varepsilon(t)$	$\frac{1}{s+a}$	$\text{Re}\{s\}>\text{Re}\{-a\}$
$-e^{-at}\varepsilon(-t)$	$\frac{1}{s+a}$	$\text{Re}\{s\}<\text{Re}\{-a\}$
$t\varepsilon(t)$	$\frac{1}{s^2}$	$\text{Re}\{s\}>0$
$t^n\varepsilon(t)$	$\frac{n!}{s^{n+1}}$	$\text{Re}\{s\}>0$
$te^{-at}\varepsilon(t)$	$\frac{1}{(s+a)^2}$	$\text{Re}\{s\}>\text{Re}\{-a\}$
$t^ne^{-at}\varepsilon(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}\{s\}>\text{Re}\{-a\}$
$\sin(\omega_0t)\varepsilon(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\text{Re}\{s\}>0$
$\cos(\omega_0t)\varepsilon(t)$	$\frac{s}{s^2+\omega_0^2}$	$\text{Re}\{s\}>0$
$e^{-at}\cos(\omega_0t)\varepsilon(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\text{Re}\{s\}>\text{Re}\{-a\}$
$e^{-at}\sin(\omega_0t)\varepsilon(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\text{Re}\{s\}>\text{Re}\{-a\}$
$t\cos(\omega_0t)\varepsilon(t)$	$\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$	$\text{Re}\{s\}>0$
$t\sin(\omega_0t)\varepsilon(t)$	$\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$	$\text{Re}\{s\}>0$

Appendix B.2 Properties of the Bilateral Laplace Transform

$x(t)$	$X(s)=\mathcal{L}\{x(t)\}$	ROC
Linearity $Ax_1(t)+Bx_2(t)$	$AX_1(s)+BX_2(s)$	ROC $\supseteq$ ROC$\{X_1\}$ $\cap$ROC$\{X_2\}$
Delay $x(t-\tau)$	$e^{-s\tau}X(s)$	not affected
Modulation $e^{at}x(t)$	$X(s-a)$	Re$\{a\}$ shifted by Re$\{a\}$ to the right
'Multiplication by t', Differentiation in the frequency domain $tx(t)$	$-\frac{d}{ds}X(s)$	not affected
Differentiation in the time domain $\frac{d}{dt}x(t)$	$sX(s)$	ROC $\supseteq$ ROC$\{X\}$
Integration $\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{s}X(s)$	ROC $\supseteq$ ROC$\{X\}$ $\cap\{s:\text{Re}\{s\}>0\}$
Scaling $x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	ROC scaled by a factor of a

Appendix B.3 Fourier Transform Pairs

$x(t)$	$X(j\omega)=\mathcal{F}\{x(t)\}$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\dot\delta(t)$	$j\omega$
$\frac{1}{T}\perp\!\!\perp\!\!\perp\left(\frac{t}{T}\right)$	$\perp\!\!\perp\!\!\perp\left(\frac{\omega T}{2\pi}\right)$
$\varepsilon(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$\text{rect}(at)$	$\frac{1}{\lvert a\rvert}\text{si}\left(\frac{\omega}{2a}\right)$
$\text{si}(at)$	$\frac{\pi}{\lvert a\rvert}\text{rect}\left(\frac{\omega}{2a}\right)$
$\frac{1}{t}$	$-j\pi\text{sign}(\omega)$
$\text{sign}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0t)$	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$
$\sin(\omega_0t)$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$
$e^{-\alpha\lvert t\rvert},\ \alpha>0$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$e^{-a^2t^2}$	$\frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}$

Appendix B.4 Properties of the Fourier Transform

	$x(t)$	$X(j\omega)=\mathcal{F}\{x(t)\}$
Linearity	$Ax_1(t)+Bx_2(t)$	$AX_1(j\omega)+BX_2(j\omega)$
Delay	$x(t-\tau)$	$e^{-j\omega\tau}X(j\omega)$
Modulation	$e^{j\omega_0t}x(t)$	$X(j(\omega-\omega_0))$
'Multiplication by t' Differentiation in the frequency domain	$tx(t)$	$-\frac{dX(j\omega)}{d(j\omega)}$
Differentiation in the time domain	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$X(j\omega)\left[\pi\delta(\omega)+\frac{1}{j\omega}\right]$ $=\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right),\quad a\in\mathbb{R}\backslash\{0\}$
Convolution	$x_1(t)*x_2(t)$	$X_1(j\omega)\cdot X_2(j\omega)$
Multiplication	$x_1(t)\cdot x_2(t)$	$\frac{1}{2\pi}X_1(j\omega)*X_2(j\omega)$
Duality	$x_1(t)$ $x_2(jt)$	$x_2(j\omega)$ $2\pi x_1(-\omega)$
Symmetry relations	$x(-t)$ $x^*(t)$ $x^*(-t)$	$X(-j\omega)$ $X^*(-j\omega)$ $X^*(j\omega)$
Parseval theorem	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2d\omega$

Appendix B.6 Properties of the z-Transform

Property	$x[k]$	$X(z)$	ROC
Linearity	$ax_1[k]+bx_2[k]$	$aX_1(z)+bX_2(z)$	ROC $\supseteq$ ROC$\{X_1\}\cap$ROC$\{X_2\}$
Delay	$x[k-\kappa]$	$z^{-\kappa}X(z)$	ROC$\{x\}$; separate consideration of $z=0$ and $z\to\infty$
Modulation	$a^k x[k]$	$X\left(\frac{z}{a}\right)$	ROC$=\left\{z \,\middle\|\, \frac{z}{a} \in\text{ROC}\{x\}\right\}$
Multiplication by k	$kx[k]$	$-z\frac{dX(z)}{dz}$	ROC$\{x\}$; separate consideration of $z=0$
Time inversion	$x[-k]$	$X(z^{-1})$	ROC$=\{z \,\|\, z^{-1}\in\text{ROC}\{x\}\}$
Convolution	$x_1[k]*x_2[k]$	$X_1(z)\cdot X_2(z)$	ROC $\supseteq$ ROC$\{x_1\}\cap$ROC$\{x_2\}$
Multiplication	$x_1[k]\cdot x_2[k]$	$\frac{1}{2\pi j}\oint X_1(\zeta)X_2\left(\frac{z}{\zeta}\right)\frac{1}{\zeta}d\zeta$	multiply the limits of the ROC

Appendix B.5 Two-sided z-Transform Pairs

$x[k]$	$X(z)=\mathcal{Z}\{x[k]\}$	ROC
$\delta[k]$	1	$z\in\mathbb{C}$
$\varepsilon[k]$	$\frac{z}{z-1}$	$\|z\|>1$
$a^k\varepsilon[k]$	$\frac{z}{z-a}$	$\|z\|>\|a\|$
$-a^k\varepsilon[-k-1]$	$\frac{z}{z-a}$	$\|z\|<\|a\|$
$k\varepsilon[k]$	$\frac{z}{(z-1)^2}$	$\|z\|>1$
$ka^k\varepsilon[k]$	$\frac{az}{(z-a)^2}$	$\|z\|>\|a\|$
$\sin(\Omega_0 k)\varepsilon[k]$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$	$\|z\|>1$
$\cos(\Omega_0 k)\varepsilon[k]$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$	$\|z\|>1$