NTNU – Trondheim
Norwegian University of
Science and Technology

Examination paper for TFY4280 Signal Processing

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Examination date: 06.06.2014

Examination time (from-to): 0900 - 1300

Permitted examination support material:

- Simple calculator (according to NTNU exam regulations)
- K. Rottmann: Matematisk formelsamling (eller tilsvarende)
- Barnett and Cronin: Mathematical formulae
- Carl Angell og Bjørn Ebbe Lian: Fysiske størrelser og enheter, navn og symboler (eller tilsvarende)

Other information:

- Languages: English/Bokmål/Nynorsk
- Number of pages (including this page and attachments): 12
- Answer must be written in English or Norwegian. Number of points given to each sub-question is given in bold font. The maximum score for the exam is **100p**.
- **Attachment:** 2 pages with transform tables and properties.

Checked by:

Date: **Signature:**

Q1: Impulse response (30p)

A) (10p) Explain what is described by the concept of *impulse response*. For continuous-time or discrete-time LTI systems, how can *impulse response* be used to determine the output signal $y(t)$ from an arbitrary input signal $x(t)$. Consider both time and frequency domains. Are there any necessary assumptions? Explain.

B) (10p) Find impulse response function for LTI system described by differential equation given below.

$$2y(t) + 3\dot{y}(t) + \ddot{y}(t) = x(t) + 2\dot{x}(t) \tag{1}$$

C) (10p) Find impulse response function for a system described by a difference equation given below.

$$9y[n] - 9y[n-1] + 2y[n-2] = x[n] - 2x[n-1] \tag{2}$$

Q2: Frequency response (20p)

A) (10p) What is described by "frequency response" $H(j\omega)$ of a LTI system? Are initial conditions important to determine frequency response?

B) (10p) Determine expression describing $H(j\omega)$ for system described by the differential equation given in question 1B. Calculate for $\omega = 0$Hz and 10Hz.

$$2y(t) + 3\dot{y}(t) + \ddot{y}(t) = x(t) + 2\dot{x}(t) \tag{3}$$

Q3 (30p) Consider a periodic square wave signal $x(t)$ shown in Figure 1, assume that this function is defined for $-\infty < t < \infty$

A) (10p) Derive a general expression for Fourier transform of a periodic function $x_p(t)$ and Fourier transform of a sampled function $x_s(t)$ (defined in discrete time domain).

HINT: These equations should be helpful :

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \tag{4}$$

$$\mathcal{F}\left\{\delta_T(t)\right\} = \sum_{n=-\infty}^{\infty} \omega_0 \delta(\omega - n\omega_0) \tag{5}$$

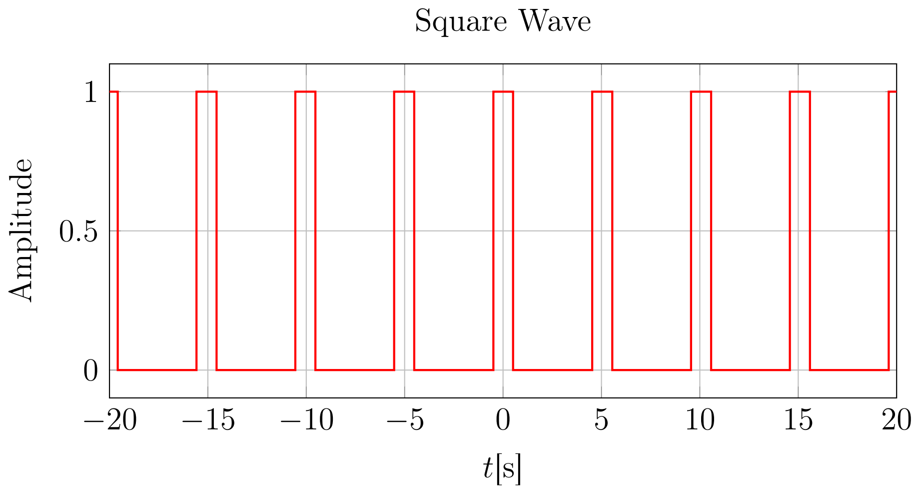

Figure 1: Square wave with repeat period of 5s and 1s pulse duration. Question 3.

B) **(10p)** Find the expression for the Fourier transform of the signal shown in Figure 1 and sketch it for $-2\pi < \omega < 2\pi$.

C) **(10p)**Define *power density spectrum* and explain how to calculate it for signal shown in Figure 1.
HINT: These equations might be helpful:

$$\Lambda\left[\frac{t}{T}\right] = \begin{cases} 1-\frac{|t|}{T} & \text{for} \quad |t|<T \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}\left\{\Lambda\left[\frac{t}{T}\right]\right\} = \frac{\sin^2(\omega T/2)}{(\omega T/2)^2}$$

Q4 **(20p)** Consider a system with impulse response h(t) $= e^t\epsilon(t)$ $(t \geq 0)$ where $\epsilon(t)$ is the unit step function.

A) **(10p)**Is the system BIBO stable?

B) **(10p)** Now put this system into the feed-back system shown in Figure 2 and find the range of A-values so that the system is BIBO stable.

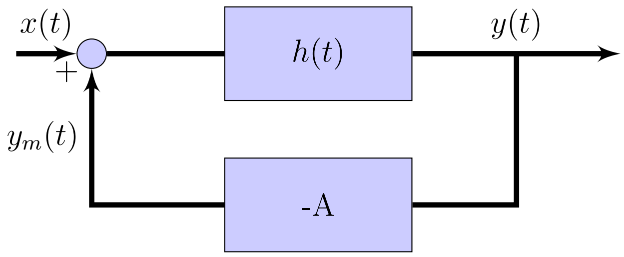

Figure 2: Feed-back system for Question 4B

Q1: Impulsrespons (30p)

A) (10p) Forklar hva som er beskrevet av konseptet *impulsrespons*. For kontinuerlige eller tidsdiskrete LTI-systemer, hvordan kan em impulsrespons brukes til å bestemme utgangssignalet $y(t)$ fra et vilkårlig inngangssignal $x(t)$. Betrakt både tids- og frekvensdomener. Er det noen nødvendige antakelser? Forklar.

B) (10p) Finn impulsresponsen for LTI-systemet beskrevet av differensialligning gitt nedenfor.

$$2y(t) + 3\dot{y}(t) + \ddot{y}(t) = x(t) + 2\dot{x}(t) \tag{6}$$

C) (10p) Finn impulsresponsfunksjonen for et system som beskrives av differanseligningen gitt nedenfor.

$$9y[n] - 9y[n-1] + 2y[n-2] = x[n] - 2x[n-1] \tag{7}$$

Q2: Frekvensrespons (20p)

A) (10p) Hva er beskrevet av "frekvensrespons" $H(j\omega)$ for et LTI-system? Er initialtilstanden ("initial conditions") viktig for å bestemme frekvensrespons?

B) (10p) Bestem uttrykket som beskriver $H(j\omega)$ for systemet beskrevet av differensialligningen gitt i spørsmålet 1B. Beregn for $\omega = 0$ Hz og 10 Hz.

$$2y(t) + 3\dot{y}(t) + \ddot{y}(t) = x(t) + 2\dot{x}(t) \tag{8}$$

Q3 (30p) Betrakt et periodisk firkantbølgesignal $x(t)$ vist i figur 3. Anta at denne funksjonen er definert for $-\infty < t < \infty$

A) (10p) Utled et generelt uttrykk for Fouriertransformen av en periodisk funksjon $x_p(t)$ og Fouriertransformen av en samplet funksjon $x_s(t)$ (funksjon som er definert i diskret tidsdomene).

Tips: Disse ligningene bør være nyttige:

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \tag{9}$$

$$\mathcal{F}\{\delta_T(t)\} = \sum_{n=-\infty}^{\infty} \omega_0 \delta(\omega - n\omega_0) \tag{10}$$

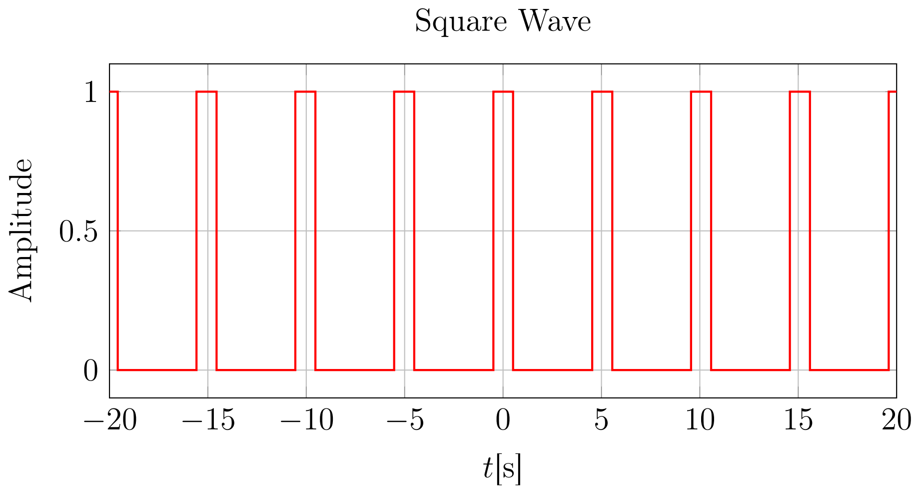

Figure 3: Firkantbølge med gjentakelsesperiode på 5s og 1s pulsvarighet. Spørsmål 3.

B) (10p) Finn uttrykket for Fouriertransformen av signalet vist i Figur 3 og skisser det for $-2\pi < \omega < 2\pi$.

C) (10p) Definer *effekttetthesspektrumet* (*power density spectrum*) og forklar hvordan man regner det for signalet vist i figur 3.
Tips: Disse ligningene kan være nyttige:

$$\Lambda\left[\frac{t}{T}\right] = \begin{cases} 1 - \frac{|t|}{T} & \text{for} \quad |t| < T \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}\left\{\Lambda\left[\frac{t}{T}\right]\right\} = \frac{\sin^2(\omega T/2)}{(\omega T/2)^2}$$

Q4 (20p) Betrakt et system med impulsresponsen $h(t) = e^t \epsilon(t)$ $(t \geq 0)$, hvor $\epsilon(t)$ er "unit step function".

A) (10p) Er systemet BIBO stabilt?

B) (10p) Sett dette systemet i feed-back systemet vist i figur 4 og finn uttrykket for A, slik at systemet er BIBO stabilt.

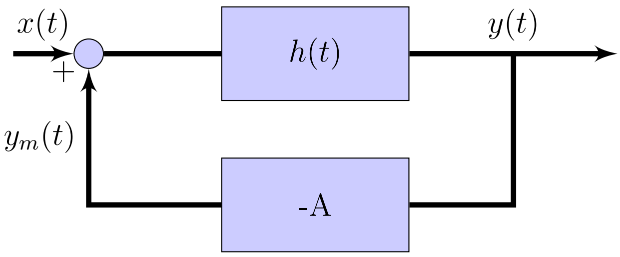

Figure 4: Feed-back system for Spørsmål 4B

Q1: Impulsrespons (30p)

A) (10p) Forklar kva som er beskreve av konseptet *impulsrespons*. For kontinuerlege eller tidsdiskrete LTI-systemer, korleis kan em impulsrespons brukas til å fastsetje utgångssignalet $y(t)$ frå eit vilkårleg inngongssignal $x(t)$. Betrakt både tids- og frekvensdomene. Er det nokre nødvendige føresetnader? Forklar.

B) (10p) Finn impulsrespons for eit LTI-system beskrive av differensiallikninga gitt under.

$$2y(t) + 3\dot{y}(t) + \ddot{y}(t) = x(t) + 2\dot{x}(t) \tag{11}$$

C) (10p) Finn impulsresponsfunksjonen for eit system som beskrivas av differanselikninga gitt under.

$$9y[n] - 9y[n-1] + 2y[n-2] = x[n] - 2x[n-1] \tag{12}$$

Q2: Frekvensrespons (20p)

A) (10p) Kva er beskrevet av "frekvensrespons" $H(j\omega)$ for eit LTI-system? Er "initialtilstanden" ("initial conditions") viktig for å bestemme frekvensrespons?

B) (10p) Bestem uttrykket som beskriv $H(j\omega)$ for systemet beskreve av differensiallikninga gitt i spørsmål 1B. Berekn for $\omega = 0$ Hz og 10 Hz.

$$2y(t) + 3\dot{y}(t) + \ddot{y}(t) = x(t) + 2\dot{x}(t) \tag{13}$$

Q3 (30p) Betrakt det periodiske firkantbølgjesignalet $x(t)$ vist i figur 5. Anta at denne funksjonen er definert for $-\infty < t < \infty$

A) (10p) Utled eit generelt uttrykk for Fouriertransformen av ein periodisk funksjon $x_p(t)$ og Fouriertransformen av ein samplet funksjon $x_s(t)$ (funksjon som er definert i diskret tidsdomene).

Tips: Disse likningane bør vere nyttige:

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \tag{14}$$

$$\mathcal{F}\{\delta_T(t)\} = \sum_{n=-\infty}^{\infty} \omega_0 \delta(\omega - n\omega_0) \tag{15}$$

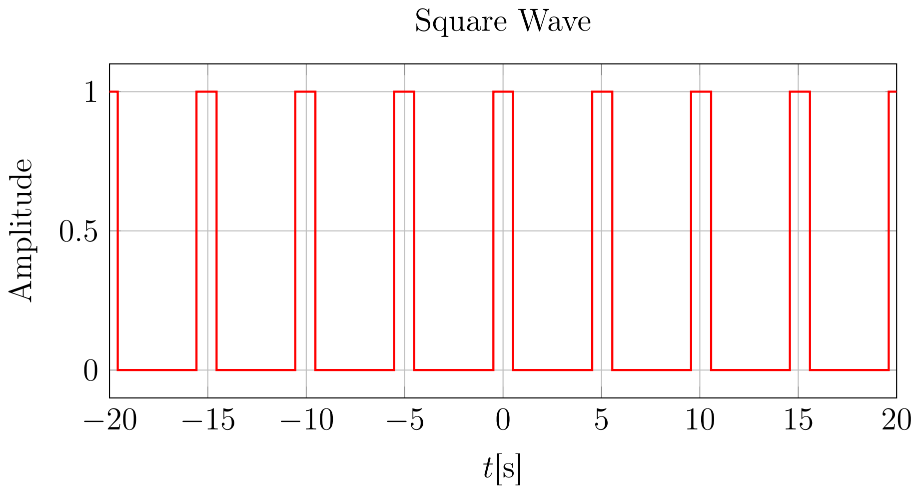

Figure 5: Firkantbølgje med gjentakingssperiode på 5s og 1s pulstid. Spørsmål 3.

- **B) (10p)** Finn uttrykket for Fouriertransformen av signalet vist i Figur 5 og skisser det for $-2\pi < \omega < 2\pi$.
- **C) (10p)** Definer *power density spectrum* og forklar korleis ein reknar det ut for signalet vist i figur 5.
  Tips: Disse likningane kan vere nyttige:

$$\Lambda\left[\frac{t}{T}\right] = \begin{cases} 1-\frac{|t|}{T} & \text{for} \quad |t|<T \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}\left\{\Lambda\left[\frac{t}{T}\right]\right\} = \frac{\sin^2(\omega T/2)}{(\omega T/2)^2}$$

Q4 (20p) Betrakt eit system med impulsresponsen $h(t) = e^t\epsilon(t)$ $(t \geq 0)$, kor $\epsilon(t)$ er "unit step function".

- **A) (10p)** Er systemet BIBO stabilt?
- **B) (10p)** Sett dette systemet i feed-back systemet vist i figur 6 og finn uttrykket for A, slik at systemet er BIBO stabilt.

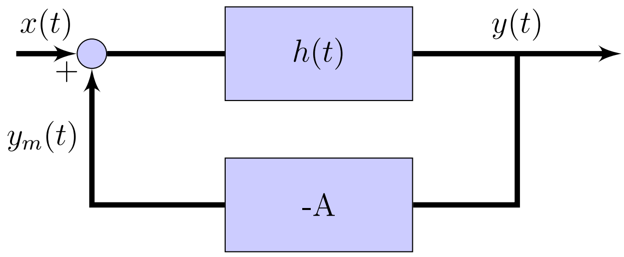

Figure 6: Feed-back system for Spørsmål 4B.

Appendix B.1 Bilateral Laplace Transform Pairs

$x(t)$	$X(s) = \mathcal{L}\{x(t)\}$	ROC
$\delta(t)$	1	$s \in \mathbb{C}$
$\varepsilon(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
$e^{-at}\varepsilon(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$-e^{-at}\varepsilon(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} < \mathrm{Re}\{-a\}$
$t\varepsilon(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$
$t^n\varepsilon(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$
$te^{-at}\varepsilon(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$t^n e^{-at}\varepsilon(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
$\cos(\omega_0 t)\varepsilon(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\varepsilon(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$e^{-at}\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$t\cos(\omega_0 t)\varepsilon(t)$	$\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\} > 0$
$t\sin(\omega_0 t)\varepsilon(t)$	$\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\} > 0$

Appendix B.2 Properties of the Bilateral Laplace Transform

	$x(t)$	$X(s) = \mathcal{L}\{x(t)\}$	ROC
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(s) + BX_2(s)$	ROC $\supseteq$ ROC$\{X_1\}$ $\cap$ROC$\{X_2\}$
Delay	$x(t-\tau)$	$e^{-s\tau}X(s)$	not affected
Modulation	$e^{at}x(t)$	$X(s-a)$	$\mathrm{Re}\{a\}$ shifted by $\mathrm{Re}\{a\}$ to the right
'Multiplication by t', Differentiation in the frequency domain	$tx(t)$	$-\frac{d}{ds}X(s)$	not affected
Differentiation in the time domain	$\frac{d}{dt}x(t)$	$sX(s)$	ROC $\supseteq$ ROC$\{X\}$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	ROC $\supseteq$ ROC$\{X\}$ $\cap\{s : \mathrm{Re}\{s\} > 0\}$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	ROC scaled by a factor of a

Appendix B.3 Fourier Transform Pairs

$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\dot{\delta}(t)$	$j\omega$
$\frac{1}{T}\text{Ш}\left(\frac{t}{T}\right)$	$\text{Ш}\left(\frac{\omega T}{2\pi}\right)$
$\varepsilon(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\mathrm{rect}(at)$	$\frac{1}{\lvert a\rvert}\mathrm{si}\left(\frac{\omega}{2a}\right)$
$\mathrm{si}(at)$	$\frac{\pi}{\lvert a\rvert}\mathrm{rect}\left(\frac{\omega}{2a}\right)$
$\frac{1}{t}$	$-j\pi\,\mathrm{sign}(\omega)$
$\mathrm{sign}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$
$\sin(\omega_0 t)$	$j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$
$e^{-\alpha\lvert t\rvert},\ \alpha > 0$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$e^{-a^2t^2}$	$\frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}$

Appendix B.4 Properties of the Fourier Transform

	$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(j\omega) + BX_2(j\omega)$
Delay	$x(t-\tau)$	$e^{-j\omega\tau}X(j\omega)$
Modulation	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
'Multiplication by t' Differentiation in the frequency domain	$tx(t)$	$-\frac{dX(j\omega)}{d(j\omega)}$
Differentiation in the time domain	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$X(j\omega)\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$ $= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right),\quad a \in \mathbb{R}\backslash\{0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega)\cdot X_2(j\omega)$
Multiplication	$x_1(t)\cdot x_2(t)$	$\frac{1}{2\pi}X_1(j\omega) * X_2(j\omega)$
Duality	$x_1(t)$ $x_2(jt)$	$x_2(j\omega)$ $2\pi x_1(-\omega)$
Symmetry relations	$x(-t)$ $x^*(t)$ $x^*(-t)$	$X(-j\omega)$ $X^*(-j\omega)$ $X^*(j\omega)$
Parseval theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$

Appendix B.6 Properties of the z-Transform

Property	$x[k]$	$X(z)$	ROC
Linearity	$ax_1[k]+bx_2[k]$	$aX_1(z)+bX_2(z)$	ROC $\supseteq$ ROC$\{X_1\}\cap$ROC$\{X_2\}$
Delay	$x[k-\kappa]$	$z^{-\kappa}X(z)$	ROC$\{x\}$; separate consideration of $z=0$ and $z\to\infty$
Modulation	$a^k x[k]$	$X\left(\frac{z}{a}\right)$	ROC$=\left\{z \,\middle\vert\, \frac{z}{a}\in\text{ROC}\{x\}\right\}$
Multiplication by k	$kx[k]$	$-z\frac{dX(z)}{dz}$	ROC$\{x\}$; separate consideration of $z=0$
Time inversion	$x[-k]$	$X(z^{-1})$	ROC$=\{z \,\vert\, z^{-1}\in\text{ROC}\{x\}\}$
Convolution	$x_1[k]*x_2[k]$	$X_1(z)\cdot X_2(z)$	ROC $\supseteq$ ROC$\{x_1\}\cap$ROC$\{x_2\}$
Multiplication	$x_1[k]\cdot x_2[k]$	$\frac{1}{2\pi j}\oint X_1(\zeta)X_2\left(\frac{z}{\zeta}\right)\frac{1}{\zeta}d\zeta$	multiply the limits of the ROC

Appendix B.5 Two-sided z-Transform Pairs

$x[k]$	$X(z)=\mathcal{Z}\{x[k]\}$	ROC
$\delta[k]$	1	$z\in\mathbb{C}$
$\varepsilon[k]$	$\frac{z}{z-1}$	$\vert z\vert>1$
$a^k\varepsilon[k]$	$\frac{z}{z-a}$	$\vert z\vert>\vert a\vert$
$-a^k\varepsilon[-k-1]$	$\frac{z}{z-a}$	$\vert z\vert<\vert a\vert$
$k\varepsilon[k]$	$\frac{z}{(z-1)^2}$	$\vert z\vert>1$
$ka^k\varepsilon[k]$	$\frac{az}{(z-a)^2}$	$\vert z\vert>\vert a\vert$
$\sin(\Omega_0 k)\varepsilon[k]$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$	$\vert z\vert>1$
$\cos(\Omega_0 k)\varepsilon[k]$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$	$\vert z\vert>1$