NTNU – Trondheim
Norwegian University of
Science and Technology

Examination paper for TFY4280 Signal Processing

Academic contact during examination: Pawel Sikorski

Phone: 98486426

Examination date: 27.05.2016

Examination time (from-to): 0900 - 1300

Permitted examination support material:

- Simple calculator (according to NTNU exam regulations)
- K. Rottmann: Matematisk formelsamling
- Barnett and Cronin: Mathematical formulae
- Carl Angell og Bjørn Ebbe Lian: Fysiske størrelser og enheter, navn og symboler

Other information about the examp paper:

- Language: English
- Number of pages (including this page and attachments): 5
- Answer must be written in English or Norwegian. Number of points given to each sub-question is given in bold font. The maximum score for the exam is **100p**.
- **Attachment:** 2 pages with transform tables and properties.

Checked by:

Date: **Signature:**

Q1. (30p) System $S_1\{\ \}$ is described by a transfer function $H_1(s)$ given below.

$$H_1(s) = \frac{1}{s^2 + 3s + 2}$$

A. Calculate unit step response for this system

B. Design a discrete-time system which is equivalent to the system $S_1\{\ \}$ studied above. Find discrete time transfer function $H(z)$ and again calculate output if a discrete time unit step function $u[n]$ is given as an input. If necessary use $t_s = 1$s and calculate only the few first terms of the output signal ($0 \leq n < 3$).

Hint: to save time, use a difference equation for the system and calculate the unit step response in the time domain.

Q2. (10p) Consider a discrete-time LTI system with a impulse response $h[n]$ given by:

$$h[n] = (-\alpha)^n \, u[n]$$

where $u[n]$ is the unit step function.

A. Is this system causal?

B. For what range of α-values is this system BIBO stable?

Q3. (10p) Find the discrete-time Fourier transform (DTFT) of the rectangular pulse sequence given by

$$x[n] = u[n] - u[n - N]$$

where $u[n]$ is the unit step function. This discrete-time signal is sampled with a sampling frequency ω_s, write the expression for the transform both in the discrete frequency domain and in the frequency domain.

Note: it is enough to write the aswer as a fraction of two complex functions and you do not need to arrive at an elegant expression.

Q4. (10p) Find the inverse z-transform of

$$X(z) = z^2 \left(1 - \frac{1}{2}z^{-1}\right) \left(1 - z^{-1}\right) \left(1 + 2z^{-1}\right) \quad 0 < |z| < \infty$$

Q5. (10p) Find the z-transform of the following signal

$$x[n] = a^{-n}u[-n]$$

Q6. (20p) If the Fourier transform of a signal $x(t)$ is given by $X(\omega)$, find an expression for the Fourier transform $X_s(\omega)$ of signal $x_s(t)$ defined as:

$$x_s(t) = \delta_T(t)x(t)$$

where

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kt_s)$$

Explain how obtained expression relates to Nyquist sampling rate condition.

HINT: expression for the Fourier transform of a periodic function might be useful here:

$$F_p(\omega) = \sum_{n=-\infty}^{\infty} \omega_0 F(n\omega_0)\delta(\omega - n\omega_0)$$

Q7. (10p) What is defined by a power density spectrum of a random signal and how can it be calculated? Sketch power density spectrum of *white noise* and *band-limited white noise* signals.

Appendix B.1 Bilateral Laplace Transform Pairs

$x(t)$	$X(s) = \mathcal{L}\{x(t)\}$	ROC
$\delta(t)$	1	$s \in \mathbb{C}$
$\varepsilon(t)$	$\frac{1}{s}$	$\text{Re}\{s\} > 0$
$e^{-at}\varepsilon(t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$-e^{-at}\varepsilon(-t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} < \text{Re}\{-a\}$
$t\varepsilon(t)$	$\frac{1}{s^2}$	$\text{Re}\{s\} > 0$
$t^n\varepsilon(t)$	$\frac{n!}{s^{n+1}}$	$\text{Re}\{s\} > 0$
$te^{-at}\varepsilon(t)$	$\frac{1}{(s+a)^2}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$t^n e^{-at}\varepsilon(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\text{Re}\{s\} > 0$
$\cos(\omega_0 t)\varepsilon(t)$	$\frac{s}{s^2+\omega_0^2}$	$\text{Re}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\varepsilon(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$e^{-at}\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\text{Re}\{s\} > \text{Re}\{-a\}$
$t\cos(\omega_0 t)\varepsilon(t)$	$\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$	$\text{Re}\{s\} > 0$
$t\sin(\omega_0 t)\varepsilon(t)$	$\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$	$\text{Re}\{s\} > 0$

Appendix B.2 Properties of the Bilateral Laplace Transform

	$x(t)$	$X(s) = \mathcal{L}\{x(t)\}$	ROC
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(s) + BX_2(s)$	ROC $\supseteq$ ROC$\{X_1\}$ $\cap$ROC$\{X_2\}$
Delay	$x(t-\tau)$	$e^{-s\tau}X(s)$	not affected
Modulation	$e^{at}x(t)$	$X(s-a)$	$\text{Re}\{a\}$ shifted by $\text{Re}\{a\}$ to the right
'Multiplication by t', Differentiation in the frequency domain	$tx(t)$	$-\frac{d}{ds}X(s)$	not affected
Differentiation in the time domain	$\frac{d}{dt}x(t)$	$sX(s)$	ROC $\supseteq$ ROC$\{X\}$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	ROC $\supseteq$ ROC$\{X\}$ $\cap\{s : \text{Re}\{s\} > 0\}$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	ROC scaled by a factor of a

Appendix B.3 Fourier Transform Pairs

$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\dot{\delta}(t)$	$j\omega$
$\frac{1}{T}⊥⊥⊥\left(\frac{t}{T}\right)$	$⊥⊥⊥\left(\frac{\omega T}{2\pi}\right)$
$\varepsilon(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\text{rect}(at)$	$\frac{1}{\lvert a\rvert}\text{si}\left(\frac{\omega}{2a}\right)$
$\text{si}(at)$	$\frac{\pi}{\lvert a\rvert}\text{rect}\left(\frac{\omega}{2a}\right)$
$\frac{1}{t}$	$-j\pi\,\text{sign}(\omega)$
$\text{sign}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$
$\sin(\omega_0 t)$	$j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$
$e^{-\alpha\lvert t\rvert},\ \alpha > 0$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$e^{-a^2t^2}$	$\frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}$

Appendix B.4 Properties of the Fourier Transform

	$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(j\omega) + BX_2(j\omega)$
Delay	$x(t-\tau)$	$e^{-j\omega\tau}X(j\omega)$
Modulation	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
'Multiplication by t' Differentiation in the frequency domain	$tx(t)$	$-\frac{dX(j\omega)}{d(j\omega)}$
Differentiation in the time domain	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$X(j\omega)\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$ $= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right),\quad a \in \mathbb{R}\backslash\{0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega)\cdot X_2(j\omega)$
Multiplication	$x_1(t)\cdot x_2(t)$	$\frac{1}{2\pi}X_1(j\omega) * X_2(j\omega)$
Duality	$x_1(t)$ $x_2(jt)$	$x_2(j\omega)$ $2\pi x_1(-\omega)$
Symmetry relations	$x(-t)$ $x^*(t)$ $x^*(-t)$	$X(-j\omega)$ $X^*(-j\omega)$ $X^*(j\omega)$
Parseval theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$

Appendix B.6 Properties of the z-Transform

Property	$x[k]$	$X(z)$	ROC
Linearity	$ax_1[k]+bx_2[k]$	$aX_1(z)+bX_2(z)$	ROC $\supseteq$ ROC$\{X_1\}\cap$ROC$\{X_2\}$
Delay	$x[k-\kappa]$	$z^{-\kappa}X(z)$	ROC$\{x\}$; separate consideration of $z=0$ and $z\to\infty$
Modulation	$a^k x[k]$	$X\left(\frac{z}{a}\right)$	ROC$=\left\{z \left\| \frac{z}{a} \in\right.\text{ROC}\{x\}\right\}$
Multiplication by k	$kx[k]$	$-z\frac{dX(z)}{dz}$	ROC$\{x\}$; separate consideration of $z=0$
Time inversion	$x[-k]$	$X(z^{-1})$	ROC$=\{z \,\|z^{-1}\in$ROC$\{x\}\}$
Convolution	$x_1[k]*x_2[k]$	$X_1(z)\cdot X_2(z)$	ROC $\supseteq$ ROC$\{x_1\}\cap$ROC$\{x_2\}$
Multiplication	$x_1[k]\cdot x_2[k]$	$\frac{1}{2\pi j}\oint X_1(\zeta)X_2\left(\frac{z}{\zeta}\right)\frac{1}{\zeta}d\zeta$	multiply the limits of the ROC

Appendix B.5 Two-sided z-Transform Pairs

$x[k]$	$X(z)=\mathcal{Z}\{x[k]\}$	ROC
$\delta[k]$	1	$z\in\mathbb{C}$
$\varepsilon[k]$	$\frac{z}{z-1}$	$\|z\|>1$
$a^k\varepsilon[k]$	$\frac{z}{z-a}$	$\|z\|>\|a\|$
$-a^k\varepsilon[-k-1]$	$\frac{z}{z-a}$	$\|z\|<\|a\|$
$k\varepsilon[k]$	$\frac{z}{(z-1)^2}$	$\|z\|>1$
$ka^k\varepsilon[k]$	$\frac{az}{(z-a)^2}$	$\|z\|>\|a\|$
$\sin(\Omega_0 k)\varepsilon[k]$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$	$\|z\|>1$
$\cos(\Omega_0 k)\varepsilon[k]$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$	$\|z\|>1$