NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
Department of Physics

Contacts during the exam:
Pawel Sikorski, phone: 98486426

EXAM
TFY4335 Signal Processing

Mon 06th of June 2011. 09:00

Examination support materials:

- Formula sheet - see Appendix A
- Simple calculator (according to NTNU exam regulations)
- K. Rottmann: Matematisk formelsamling (eller tilsvarende)
- Carl Angell og Bjørn Ebbe Lian: Fysiske størrelser og enheter, navn og symboler (eller tilsvarende)

Answer must be written in English or Norwegian. Number of points given to each sub-question is given in bold font. The maximum score for the exam is **100p**. The exam consists of 7 questions: 4 questions 10 point each (section A) and 3 questions 20p each (section B).

A

maximum 10 points for each of the question.

A1 (10p) Plot auto correlation and self-convolution for each of the two signals shown in Figure 1. Both signals are defined (not zero) for t in the range $[-\pi, \pi]$.

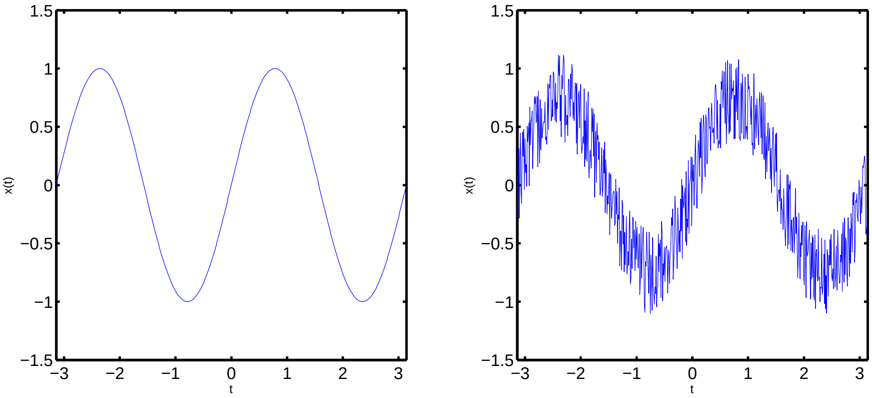


Figure 1: Question A1

ANSWER:

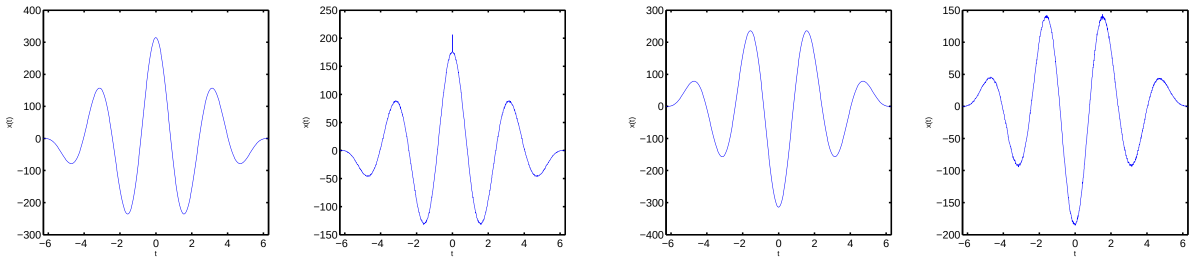


Figure 2: Answer: A1

A2 (10p) Calculate response to a unit step function input ($x(t) = \varepsilon(t)$) and delta impulse $x(t) = \delta(t)$ from a given impulse response function in the time domain:

$$h(t) = 2\varepsilon(t) - 0.5e^{-3t}\varepsilon(t)$$

ANSWER:
For $x(t) = \delta(t)$ the output is just given by h(t). For unit step function, we can calculate this from convolution between $h(t)$ and x(t) or through the use of Laplace transform. Using Laplace transform:

$$\begin{aligned}
h(t) &= 2\varepsilon(t) - 0.5e^{-3t}\varepsilon(t) \\
H(s) &= 2\frac{1}{s} - 0.5\frac{1}{s+3} \\
X(s) &= \frac{1}{s} \\
Y(s) &= \frac{1}{s}\left(2\frac{1}{s} - 0.5\frac{1}{s+3}\right) = \frac{2}{s^2} - \frac{0.5}{s(s+3)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+3} \\
Y(s) &= \frac{2}{s^2} - \frac{1/6}{s} + \frac{1/6}{s+3} \\
y(t) &= 2t\varepsilon(t) - \frac{1}{6}\varepsilon(t) + \frac{1}{6}e^{-3t}\varepsilon(t)
\end{aligned}$$

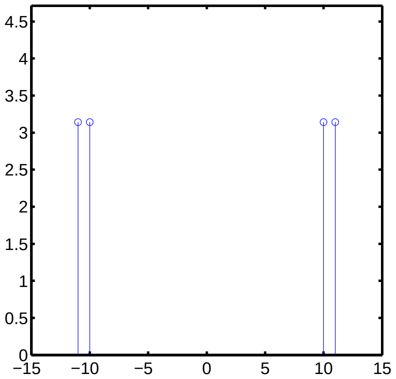
A3 (10p) Two uncorrelated processes $x(t)$ and $y(t)$ have the ensemble averages 2 and 0, respectively. Moreover, $E\{x^2(t)\} = 5$ and $E\{y^2(t)\} = 2$. Define the random process $z(t) = x(t) + y(t)$. Determine the averages: $\mu_z(t)$, $E\{z^2(t)\}$ and $\sigma_z^2(t)$

ANSWER:

$$\begin{aligned}
\mu_z &= E\{x(t)\} + E\{y(t)\} = 2 \\
E\{z^2(t)\} &= E\{[x(t) + y(t)]^2\} = E\{x^2(t) + 2x(t)y(t) + y^2(t)\} = \\
&= E\{x^2(t)\} + 2E\{x(t)y(t)\} + E\{y^2(t)\} = 5 + 0 + 2 = 7 \\
\sigma_z^2 &= E\{z^2(t)\} - \mu_z^2 = 7 - 4 = 3
\end{aligned}$$

A4 (10p) The signal $x(t)$ passes through a square law device giving output $y(t) = [x(t)]^2$

$$x(t) = \cos(10t) + \cos(11t)$$



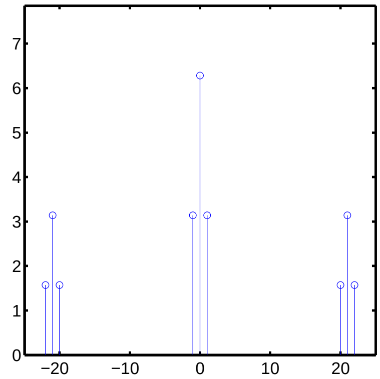


Figure 3: Answer: A4

Determine the Fourier transform of the output $\mathcal{F}\{y(t)\}$.

ANSWER:
We can use the fact that multiplication will be a convolution in the Fourier space and first calculate Fourier transform of $x(t)$

B

maximum 10 points for each of the question.

B1 (20p)

1. Show that

$$\mathcal{L}\{t \cdot x(t)\} = -\frac{\mathrm{d}X(s)}{\mathrm{d}s}$$

2. Use above property to calculate output of LTI system where input $x(t) = te^{-9t}$ defined for $t > 0$ and the impulse response is given by:

$$H(s) = \frac{1}{(s+10)}$$

ANSWER:
Using the definition of the Laplace transform:

$$\mathcal{L}\{tx(t)\} = \int_{-\infty}^{\infty} tx(t)e^{-st}\mathrm{d}t = -\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}s}\left[x(t)e^{-st}\right]\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}s}\int_{-\infty}^{\infty}\left[x(t)e^{-st}\right]\mathrm{d}t = -\frac{\mathrm{d}X(s)}{\mathrm{d}s}$$

Now we can get the Laplace transform of the input signal:

$$\mathcal{L}\left\{t \cdot e^{-at}\right\} = -\frac{\mathrm{d}\mathcal{L}\left\{e^{-at}\right\}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s+a}\right) = (s+a)^{-2}$$

$$Y(s) = \frac{1}{(s+10)} \cdot \frac{1}{(s+9)^2} = \left[\frac{A}{(s+10)} + \frac{B}{(s+9)^2} + \frac{C}{(s+9)}\right]$$

Need to solve by first letting $s = -10$ and $s = -9$:

$$\frac{1}{(9-10)^2} = A$$
$$A = 1$$
$$B = \frac{1}{-9+10} = 1/2$$
$$\frac{1}{s+10} + \frac{1}{(s+9)^2} + \frac{C}{(s+9)} = \frac{1}{(s+10)} \cdot \frac{1}{(s+9)^2}$$
$$\frac{(s+9)^2 + (s+10) + C(s+10)(s+9)}{(s+10)(s+9)^2} = \frac{1}{(s+10)(s+9)^2}$$
$$Cs^2 + s^2 = 0$$
$$C = -1$$

Check:

$$\frac{(s+9)^2 + (s+10) - (s+10)(s+9)}{(s+10)(s+9)^2} =$$
$$= \frac{(s^2 + 18s + 81 + s + 10 - s^2 - 9s - 10s - 90)}{(s+10)(s+9)^2} = \frac{1}{(s+10)(s+9)^2} = OK$$

So:

$$Y(s) = \left[\frac{1}{(s+10)} + \frac{1}{(s+9)^2} - \frac{1}{(s+9)}\right]$$
$$y(t) = e^{-10t}\varepsilon(t) + te^{-9t}\varepsilon(t) - e^{-9t}\varepsilon(t)$$

B2 (20p) Hypothetical measured signal can be represented by an analytical formula:

$$x(t) = \left[1 - e^{-t} + \sin(10t)\right]\varepsilon(t)$$

Design and draw circuit diagram for a 1-st order low pass Butterworth filter which will remove 90.9090% of the power carried by the sin part of the signal. For 1-st order low pass Butterworth filter:

$$|H(\omega)| = \frac{1}{\sqrt{1 + \frac{\omega^2}{\omega_c^2}}} \quad \omega_c = \frac{1}{RC}$$

use R = 1000Ω. Calculate how this filter will effect the exponential part of the signal, that is, calculate the output, if the input signal is given by

$$x_0(t) = \left[1 - e^{-t}\right]\varepsilon(t)$$

ANSWER:

First we need to calculate ω_c. From the exercise text we want a filter which will have $|H(\omega =$

$10)|^2 = 100\% - 90.9091\% = 0.09091$

$$|H(\omega = 10)|^2 = 0.0909090$$
$$\frac{1}{1+\frac{10^2}{\omega_c^2}} = 0.0909090$$
$$1 + \frac{10^2}{\omega_c^2} = 11$$
$$\frac{10^2}{\omega_c^2} = 10$$
$$\omega_c^2 = 10$$
$$\omega_c = \sqrt{10}$$

Now:

$$\sqrt{10} = \frac{1}{1000 \cdot C}$$
$$C = \frac{1}{1000\sqrt{10}} = 3.2 \times 10^{-4} F$$
$$RC = 0.32$$

For 1-st order low pass Butterworth circuit:

$$H(s) = \frac{1}{1+sRC}$$

Using $RC = 0.32$ and $RC^{-1} = 3.2$ we get and the Laplace transform of the input signal $x_0(t)$ is

$$X_0(s) = \mathcal{L}\{x_0(t)\} = \mathcal{L}\left\{\varepsilon(t) - e^{-t}\varepsilon(t)\right\} = \frac{1}{s} - \frac{1}{s+1}$$
$$Y(s) = X_0(s)H(s) = \left[\frac{1}{s} - \frac{1}{s+1}\right]\frac{1}{1+sRC} = \frac{1}{(1+sRC)}\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{1+sRC} + \frac{C}{s+1}$$
$$A = 1 \quad B = \frac{-RC}{-\frac{1}{RC}+1} = \frac{-0.32}{-2.2} = 0.14 \quad C = \frac{1}{RC-1} = -1.42$$
$$Y(s) = \frac{1}{s} + 0.14\frac{1}{1+0.32s} - 1.42\frac{1}{s+1}$$
$$Y(s) = \frac{1}{s} + 0.14\frac{3.2}{s+3.2} - 1.42\frac{1}{s+1}$$
$$Y(s) = \frac{1}{s} + 0.448\frac{1}{s+3.2} - 1.42\frac{1}{s+1}$$

$$Y(s) = \varepsilon(t)\left[1 + 0.448e^{-3.2t} - 1.42e^{-t}\right]$$

B3 (20p) The figure below (Figure 5) shows a digital filter in which the delays are 0.5 ms.

Write down the difference equation and from this derive the z-transfer function. Analyse the system and calculate the three first output terms ($0 \le n < 3$) for the unit step response excitation.

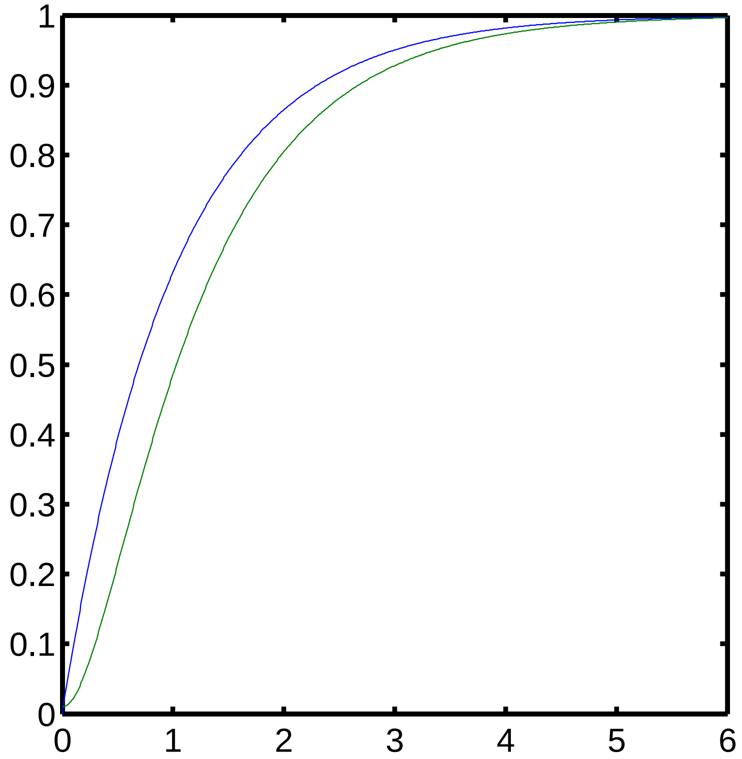


Figure 4: Answer: B2

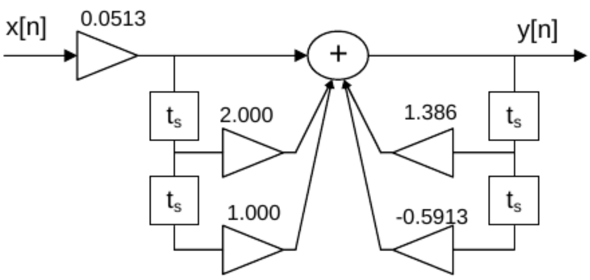


Figure 5: Question B3

ANSWER:

From the diagram:

$$y[n] = 0.0513(x[n] + 2x[n-1] + x[n-2]) + 1.386y[n-1] - 0.5913y[n-2]$$

$$Y(z) = 0.0513X(z)(1 + 2z^{-1} + z^{-2}) + Y(z)(1.386z^{-1} - 0.5913z^{-2})$$

$$Y(z)(1 - 1.386z^{-1} + 0.5913z^{-2}) = 0.0513X(z)(1 + 2z^{-1} + z^{-2})$$

$$H(z) = \frac{Y(z)}{X(z)} = 0.0513\frac{1 + 2z^{-1} + z^{-2}}{1 - 1.386z^{-1} + 0.5913z^{-2}} = 0.0513\frac{z^2 + 2z + 1}{z^2 - 1.386z + 0.5913}$$

For $X(z) = \frac{z}{z-1}$

$$Y(z) = H(z)X(z) = 0.0513\frac{z^2 + 2z + 1}{z^2 - 1.386z + 0.5913}\frac{z}{z-1}$$

$$Y(z) = 0.0513\frac{z^3 + 2z^2 + z}{(z^2 - 1.386z + 0.5913)(z-1)} =$$

$$= 0.0513\frac{z^3 + 2z^2 + z}{(z^3 - 1.386z^2 + 0.5913z - z^2 + 1.386z - 0.5913)} =$$

$$= 0.0513\frac{z^3 + 2z^2 + z}{(z^3 - 2.386z^2 + 1.9773z - 0.5913)} =$$

Output can be obtained from long division of the two fractions:

$$\begin{aligned} y[0] &= 0.0513 \\ y[1] &= 0.225 \\ y[2] &= 0.4867 \end{aligned}$$

Or by using difference equation and $x[n] = 1$ for $n \geq 0$

$$\begin{aligned} y[-2] &= 0; \\ y[-1] &= 0; \\ y[0] &= 0.0513(x[0] + 2x[-1] + x[-2]) + 1.386y[-1] - 0.5913y[-2] = 0.0513 \\ y[1] &= 0.0513(x[1] + 2x[0] + x[-1]) + 1.386y[0] - 0.5913y[-1] = \\ &= 0.0513(1 + 2 + 1.386) = 0.225 \\ y[2] &= 0.0513(x[2] + 2x[1] + x[0]) + 1.386y[1] - 0.5913y[0] = \\ &= 0.0513 \cdot 4 + 1.386 \cdot 0.225 - 0.5913 \cdot 0.0513 = 0.4867 \end{aligned}$$

Appendix B.1 Bilateral Laplace Transform Pairs

$x(t)$	$X(s)=\mathcal{L}\{x(t)\}$	ROC
$\delta(t)$	1	$s \in \mathbb{C}$
$\varepsilon(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\}>0$
$e^{-at}\varepsilon(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$-e^{-at}\varepsilon(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\}<\mathrm{Re}\{-a\}$
$t\varepsilon(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\}>0$
$t^n\varepsilon(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\}>0$
$te^{-at}\varepsilon(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$t^ne^{-at}\varepsilon(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\}>0$
$\cos(\omega_0 t)\varepsilon(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\}>0$
$e^{-at}\cos(\omega_0 t)\varepsilon(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$e^{-at}\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$t\cos(\omega_0 t)\varepsilon(t)$	$\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\}>0$
$t\sin(\omega_0 t)\varepsilon(t)$	$\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\}>0$

Appendix B.2 Properties of the Bilateral Laplace Transform

	$x(t)$	$X(s)=\mathcal{L}\{x(t)\}$	ROC
Linearity	$Ax_1(t)+Bx_2(t)$	$AX_1(s)+BX_2(s)$	ROC $\supseteq$ ROC$\{X_1\}$ $\cap$ROC$\{X_2\}$
Delay	$x(t-\tau)$	$e^{-s\tau}X(s)$	not affected
Modulation	$e^{at}x(t)$	$X(s-a)$	Re$\{a\}$ shifted by Re$\{a\}$ to the right
'Multiplication by t', Differentiation in the frequency domain	$tx(t)$	$-\frac{d}{ds}X(s)$	not affected
Differentiation in the time domain	$\frac{d}{dt}x(t)$	$sX(s)$	ROC $\supseteq$ ROC$\{X\}$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	ROC $\supseteq$ ROC$\{X\}$ $\cap\{s:\mathrm{Re}\{s\}>0\}$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	ROC scaled by a factor of a

Appendix B.3 Fourier Transform Pairs

$x(t)$	$X(j\omega)=\mathcal{F}\{x(t)\}$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\dot{\delta}(t)$	$j\omega$
$\frac{1}{T}\perp\!\!\!\perp\!\!\!\perp\left(\frac{t}{T}\right)$	$\perp\!\!\!\perp\!\!\!\perp\left(\frac{\omega T}{2\pi}\right)$
$\varepsilon(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$\mathrm{rect}(at)$	$\frac{1}{\lvert a\rvert}\mathrm{si}\left(\frac{\omega}{2a}\right)$
$\mathrm{si}(at)$	$\frac{\pi}{\lvert a\rvert}\mathrm{rect}\left(\frac{\omega}{2a}\right)$
$\frac{1}{t}$	$-j\pi\mathrm{sign}(\omega)$
$\mathrm{sign}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$
$\sin(\omega_0 t)$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$
$e^{-\alpha\lvert t\rvert},\ \alpha>0$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$e^{-a^2t^2}$	$\frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}$

Appendix B.4 Properties of the Fourier Transform

	$x(t)$	$X(j\omega)=\mathcal{F}\{x(t)\}$
Linearity	$Ax_1(t)+Bx_2(t)$	$AX_1(j\omega)+BX_2(j\omega)$
Delay	$x(t-\tau)$	$e^{-j\omega\tau}X(j\omega)$
Modulation	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
'Multiplication by t' Differentiation in the frequency domain	$tx(t)$	$-\frac{dX(j\omega)}{d(j\omega)}$
Differentiation in the time domain	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$X(j\omega)\left[\pi\delta(\omega)+\frac{1}{j\omega}\right]$ $=\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right),\quad a\in\mathbb{R}\backslash\{0\}$
Convolution	$x_1(t)*x_2(t)$	$X_1(j\omega)\cdot X_2(j\omega)$
Multiplication	$x_1(t)\cdot x_2(t)$	$\frac{1}{2\pi}X_1(j\omega)*X_2(j\omega)$
Duality	$x_1(t)$ $x_2(jt)$	$x_2(j\omega)$ $2\pi x_1(-\omega)$
Symmetry relations	$x(-t)$ $x^*(t)$ $x^*(-t)$	$X(-j\omega)$ $X^*(-j\omega)$ $X^*(j\omega)$
Parseval theorem	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2\,dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2 d\omega$

Appendix B.6 Properties of the z-Transform

Property	$x[k]$	$X(z)$	ROC
Linearity	$ax_1[k]+bx_2[k]$	$aX_1(z)+bX_2(z)$	ROC $\supseteq$ ROC$\{X_1\}\cap$ROC$\{X_2\}$
Delay	$x[k-\kappa]$	$z^{-\kappa}X(z)$	ROC$\{x\}$; separate consideration of $z=0$ and $z\to\infty$
Modulation	$a^k x[k]$	$X\left(\frac{z}{a}\right)$	ROC$=\left\{z \,\middle\|\, \frac{z}{a} \in\text{ROC}\{x\}\right\}$
Multiplication by k	$kx[k]$	$-z\frac{dX(z)}{dz}$	ROC$\{x\}$; separate consideration of $z=0$
Time inversion	$x[-k]$	$X(z^{-1})$	ROC$=\{z \,\|\, z^{-1}\in$ROC$\{x\}\}$
Convolution	$x_1[k]*x_2[k]$	$X_1(z)\cdot X_2(z)$	ROC $\supseteq$ ROC$\{x_1\}\cap$ROC$\{x_2\}$
Multiplication	$x_1[k]\cdot x_2[k]$	$\frac{1}{2\pi j}\oint X_1(\zeta)X_2\left(\frac{z}{\zeta}\right)\frac{1}{\zeta}d\zeta$	multiply the limits of the ROC

Appendix B.5 Two-sided z-Transform Pairs

$x[k]$	$X(z)=\mathcal{Z}\{x[k]\}$	ROC
$\delta[k]$	1	$z\in\mathbb{C}$
$\varepsilon[k]$	$\frac{z}{z-1}$	$\|z\|>1$
$a^k\varepsilon[k]$	$\frac{z}{z-a}$	$\|z\|>\|a\|$
$-a^k\varepsilon[-k-1]$	$\frac{z}{z-a}$	$\|z\|<\|a\|$
$k\varepsilon[k]$	$\frac{z}{(z-1)^2}$	$\|z\|>1$
$ka^k\varepsilon[k]$	$\frac{az}{(z-a)^2}$	$\|z\|>\|a\|$
$\sin(\Omega_0 k)\varepsilon[k]$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$	$\|z\|>1$
$\cos(\Omega_0 k)\varepsilon[k]$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$	$\|z\|>1$