## Q1: Impulse response (30p)

**A) (10p)** Explain what is described by the concept of *impulse response*. For continuous-time or discrete-time LTI systems, how can *impulse response* be used to determine the output signal $y(t)$ from an arbitrary input signal $x(t)$. Consider both time and frequency domains. Are there any necessary assumptions? Explain.
Impulse response: response of LTI system to an impulse input (a delta function); valid both for time continuous and time discrete systems.

$$h(t) = S\{\delta(t)\}$$

Motivation: Since $\mathcal{F}\left\{\delta(t)\right\} = 1$, all possible frequencies are represented in this input signal; and therefore, the output for such signal will probe how the system will respond to all possible frequencies (theoretically). In addition, an arbitrary input can also be described by a sum of weighed and time shifted delta impulses (convolution integral). Therefore, the output is a convolution between input and impulse response function:

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\mathrm{d}\tau$$

For this approach to be valid, convolution integral must converge. In the frequency domain, H(s) is the system transfer function,

$$H(s) = \frac{Y(s)}{X(s)}$$

therefore

$$Y(s) = H(s)X(s)$$

where

$$H(s) = \mathscr{L}\left\{h(t)\right\}.$$

**B) (10p)** Find impulse response function for LTI system described by differential equation given below.

$$2y(t) + 3\dot{y}(t) + \ddot{y}(t) = x(t) + 2\dot{x}(t) \tag{1}$$

$$
\begin{aligned}
2Y(s) + 3sY(s) + s^2Y(s) &= X(s) + 2sX(s) \\
Y(s)[2 + 3s + s^2] &= X(s)[1 + 2s] \\
H(s) = \frac{Y(s)}{H(s)} &= \frac{1 + 2s}{2 + 3s + s^2} = \frac{2s + 1}{(s + 1)(s + 2)} \\
\frac{A}{(s + 1)} + \frac{B}{(s + 2)} &= \frac{2s + 1}{(s + 1)(s + 2)} \\
A = -1 \quad B = 3 & \\
h(t) &= \epsilon(t)[3\mathrm{e}^{-2t} - \mathrm{e}^{-t}]
\end{aligned}
$$

**C) (10p)** Find impulse response function for a system described by a difference equation given below.

$$9y[n] - 9y[n-1] + 2y[n-2] = x[n] - 2x[n-1] \tag{2}$$

One can use z-transform of delta input and calculate output for this time-discrete system. Using z-transform:

$$
\begin{aligned}
9y[n] - 9y[n-1] + 2y[n-2] &= x[n] - 2x[n-1] \\
9Y(z) - 9Y(z)z^{-1} + 2Y(z)z^{-2} &= X(z) - 2X(z)z^{-1} \\
Y(z)[9 - 9z^{-1} + 2z^{-2}] &= X(z)[1 - 2z^{-1}] \\
H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 2z^{-1}}{9 - 9z^{-1} + 2z^{-2}} &= \frac{z^2 - 2z}{9z^2 - 9z + 2} = \\
= \frac{z(z-2)}{(3z-1)(3z-2)} &= \frac{1}{9}\frac{z(z-2)}{(z-1/3)(z-2/3)} = \\
= \frac{1}{9}\left[\frac{Az}{(z-1/3)} + \frac{Bz}{(z-2/3)}\right] &= \frac{1}{9}\left[\frac{5z}{(z-1/3)} - \frac{4z}{(z-2/3)}\right] = \\
= \frac{5}{9}\frac{z}{(z-1/3)} - \frac{4}{9}\frac{z}{(z-2/3)} &
\end{aligned}
$$

$$h[n] = \frac{5}{9}\left(\frac{1}{3}\right)^n u[n] - \frac{4}{9}\left(\frac{2}{3}\right)^n u[n]$$

## Q2: Frequency response (20p)

**A) (10p)** What is described by "frequency response" $H(j\omega)$ of a LTI system? Are initial conditions important to determine frequency response?

$H(j\omega)$ complex function which describes how different frequency components of the input signal will be effected by the LTI system, since in the Fourier domain

$$Y(j\omega) = X(j\omega)H(j\omega).$$

Frequency response is defined as a Fourier transform of the impulse response function or equlivelantly as system transfer function $H(s)$ calculated for $s = j\omega$:

$$H(j\omega) = H(s)|_{s=j\omega}.$$

Frequency response contain information on how both magnitude and phase of the input signal will be changed by a LTI system. $|H(j\omega)|$ (gain) describes how LTI system will change the amplitude of the input signal for frequency $\omega$. The phase angle $\arg(H(j\omega))$ will contain information on how the phase of the input signal will be effected at different frequencies. Non zero initial conditions will not effect systems frequency response, as by setting $s = j\omega$ we ignore increasing/decreasing parts of the signal and investigate system response at $t \to \infty$. Since zero-input response is transient, it will not effect systems frequency response.

**B) (10p)** Determine expression describing $H(j\omega)$ for system described by the differential equation given in question 1B. Calculate for $\omega = 0\text{Hz}$ and $10\text{Hz}$.

$$2y(t) + 3\dot{y}(t) + \ddot{y}(t) = x(t) + 2\dot{x}(t) \tag{3}$$

We alreadye know $H(s)$, and frequency response

$$\begin{aligned} H(s) = \frac{Y(s)}{H(s)} &= \frac{1+2s}{2+3s+s^2} = \frac{2s+1}{(s+1)(s+2)} = \frac{3}{(s+2)} - \frac{1}{(s+1)} \\ H(s) &= \left.\frac{3}{(s+2)} - \frac{1}{(s+1)}\right|_{s=j\omega} = 3\frac{(2-j\omega)}{(4+\omega^2)} - \frac{(1-j\omega)}{(1+\omega^2)} = \\ &= \frac{6}{4+\omega^2} - \frac{3j\omega}{4+\omega^2} - \frac{1}{1+\omega^2} + \frac{j\omega}{1+\omega^2} \end{aligned}$$

Now we need to group real and complex terms and calculate accordingly to:

$$|H(j\omega)| = \left[Re\{H(j\omega)\}^2 + Im\{H(j\omega)\}^2\right]^{1/2}$$

$$Re\{H(j\omega)\} = \frac{6}{4+\omega^2} - \frac{1}{1+\omega^2} = \frac{6+6\omega^2-4-\omega^2}{(4+\omega^2)(1+\omega^2)} = \frac{2+5\omega^2}{(4+\omega^2)(1+\omega^2)}$$
$$Im\{H(j\omega)\} = \frac{\omega}{1+\omega^2} - \frac{3\omega}{4+\omega^2} = \frac{4\omega+\omega^3-3\omega-3\omega^3}{(4+\omega^2)(1+\omega^2)} = \frac{\omega-2\omega^3}{(4+\omega^2)(1+\omega^2)}$$

Then $|H(0)| = 0.5$ and $|H(10)| = 0.1954$. Phase: $\arg(H(0)) = 0$ and $\arg(H(10)) = -76\,\text{deg}$

**Q3 (30p)** Consider a periodic square wave signal $x(t)$ shown in Figure 1, assume that this function is defined for $-\infty < t < \infty$

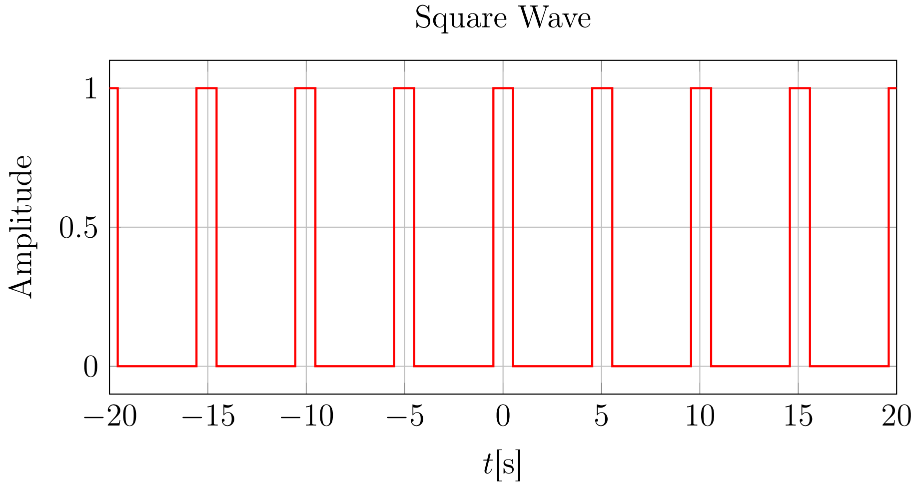



Figure 1: Square wave with repeat period of 5s and 1s pulse duration. Question 3.

**A) (10p)** Derive a general expression for Fourier transform of a periodic function $x_p(t)$ and Fourier transform of a sampled function $x_s(t)$ (defined in discrete time domain).
**HINT:** These equations should be helpful :

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \tag{4}$$
$$\mathcal{F}\{\delta_T(t)\} = \sum_{n=-\infty}^{\infty} \omega_0 \delta(\omega - n\omega_0) \tag{5}$$

We define periodic function as convolution between train of delta impulses $\delta_T(t)$ with the periodicity of the periodic function in question ($\omega_0$), and a aperiodic function $x(t)$. Convolution in the time domain corresponds to multiplication in the frequency domain. Then,

$$\begin{aligned} x_p(t) &= x(t) * \delta_T(t) \\ \mathcal{F}\{x_p(t)\} &= X_p(j\omega) = \mathcal{F}\{x(t)\}\mathcal{F}\{\delta_T(t)\} = \\ &= X(j\omega)\sum_{n=-\infty}^{\infty}\omega_0\delta(\omega - n\omega_0) = \\ &= \sum_{n=-\infty}^{\infty}\omega_0 X(n\omega_0)\delta(\omega - n\omega_0) \end{aligned}$$

In a similar manner, discrete time function can be defined as multiplication of a train of delta impulses and a time continuous signal.

$$\begin{aligned} x_s(t) &= \delta_T(t)x(t) \\ \mathcal{F}\{x_s(t)\} &= \mathcal{F}\{\delta_T(t)x(t)\} = \frac{1}{2\pi}\mathcal{F}\{\delta_T(t)\} * \mathcal{F}\{x(t)\} = \\ &= \frac{1}{2\pi}X(\omega) * \sum_{n=-\infty}^{\infty}\omega_s\delta(\omega - n\omega_s) \end{aligned}$$

Using properties of a delta function, we get

$$X_s(j\omega) = \frac{\omega_s}{2\pi}\sum_{n=-\infty}^{\infty}X(\omega - n\omega_s)$$

**B) (10p)** Find the expression for the Fourier transform of the signal shown in Figure 1 and sketch it for $-2\pi < \omega < 2\pi$. Using:

$$\mathcal{F}\{x_p(t)\} = \sum_{n=-\infty}^{\infty}\omega_0 X(n\omega_0)\delta(\omega - n\omega_0)$$

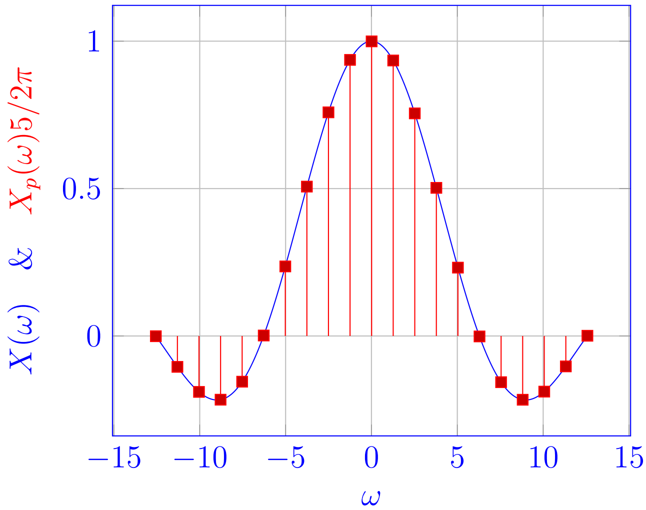
we have $\omega_0 = \frac{2\pi}{5s}$ and we only need to find FT of non-periodic function (1 period centred around zero)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t = \int_{-T/2}^{T/2}\mathrm{e}^{-j\omega t}\mathrm{d}t = \frac{1}{-j\omega}\left[\mathrm{e}^{-j\omega T/2} - \mathrm{e}^{j\omega T/2}\right] =$$

$$= T\,\mathrm{sinc}\left(\frac{\omega \mathrm{T}}{2}\right)$$

with $T = 1s$. FT of this periodic function is discrete in frequency, defined at $\omega = n\frac{2\pi}{5s}$ and has samplitude given by

$$X_p(j\omega) = \frac{2\pi}{5}\sum_{n=-\infty}^{\infty} \text{sinc}\left(\text{n}\frac{2\pi}{10}\right)\delta\left(\omega - \text{n}\frac{2\pi}{5}\right)$$




**C) (10p)** Define *power density spectrum* and explain how to calculate it for signal shown in Figure 1.
HINT: These equations might be helpful:

$$\Lambda\left[\frac{t}{T}\right] = \begin{cases} 1 - \frac{|t|}{T} & \text{for} \quad |t| < T \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}\left\{\Lambda\left[\frac{t}{T}\right]\right\} = \frac{\sin^2(\omega T/2)}{(\omega T/2)^2}$$

Few different possiblities. PSD for deterministic signal determined from Parseval's theorem (energy spectrum of the signal), or using auto correlation of the input signal, and then callculating FT. For the 2nd Method a.c.f will be deterministic, and periodic; it will be defined by triangle impulses with the same periodicity and original signal, and width of $2T$. For the signal given in the exercise text, $T = 1s$ and

$$X_p(j\omega) = \sum_{n=-\infty}^{\infty} \text{sinc}\left(\text{n}\frac{2\pi}{10}\right)\delta\left(\omega - \text{n}\frac{2\pi}{5}\right)$$

and

$$|X_p(j\omega)|^2 = \sum_{n=-\infty}^{\infty} \text{sinc}^2\left(n\frac{2\pi}{10}\right)\delta\left(\omega - n\frac{2\pi}{5}\right)$$

The same answer will can we get from using autocorrelation approach. $\varphi_x x(t)$ is a train of triangular impulses with $T = 1s$ (total width of 2s) which are periodic with the same periodicity as the original signal.

$$\Phi_{xx} = \mathcal{F}\{\varphi\} = \mathcal{F}\left\{\Lambda\left[\frac{t}{T}\right] * \delta_T(t)\right\}$$

which gives the same answer as above.

**Q4 (20p)** Consider a system with impulse response h(t) $= e^t\epsilon(t)$ $(t \geq 0)$ where $\epsilon(t)$ is the unit step function.

**A) (10p)**Is the system BIBO stable?

A requirement for a system to be BIBO stable is that

$$\int_{-\infty}^{\infty} |h(t)|dt \leq M$$

for some finite $M$. With $h(t) = e^t \cdot \epsilon(t)$ the above integral clearly does not converge and the system is not BIBO stable.

**B) (10p)** Now put this system into the feed-back system shown in Figure 2 and find the range of A-values so that the system is BIBO stable.

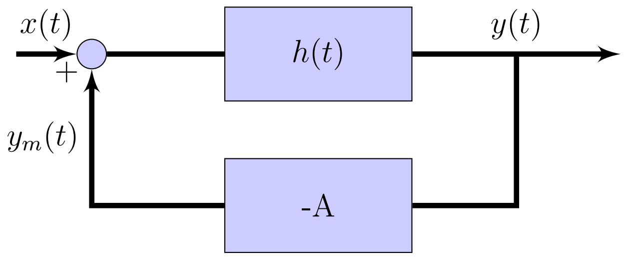



Figure 2: Feed-back system for Question 4B

Start by finding the transfer function $H(s)$

$$H(s) = \int_0^{\infty} e^t e^{-st}dt = \int_0^{\infty} e^{-(s-1)t} - \left.\frac{e^{-(s-1)t}}{s-1}\right|_{t=0}^{\infty} = \frac{1}{s-1}.$$

Note that the integral above only converges for $\mathrm{Re}(s) > 1$ (ROC consists of the complex plane to the right of $s = 1$). By examining the block diagram given in the problem text we are able to find the complete transfer function from input to output in the feedback loop. Begin by relating the input and output of the system,

$$Y(s) = H(s)\,[X(s) - A \cdot Y(s)]$$
$$Y(s)\,[1 + A \cdot H(s)] = H(s) \cdot X(s)$$

$$\frac{Y(s)}{X(s)} \equiv H_{tot}(s) = \frac{H(s)}{1 + A \cdot H(s)} = \frac{1/(s-1)}{1 + A/(s-1)} = \frac{1}{s - 1 + A}$$

We see that the transfer function $H_{tot}(s)$ has a single real pole $s_0 = (1 - A)$. In order for the system to be stable we require that the pole lie in the left half of the s-plane. This leads to the stability criterion

$$s_0 = 1 - A < 0 \Rightarrow A > 1.$$

## Appendix B.1 Bilateral Laplace Transform Pairs

| $x(t)$ | $X(s)=\mathcal{L}\{x(t)\}$ | ROC |
|---|---|---|
| $\delta(t)$ | $1$ | $s \in \mathbb{C}$ |
| $\varepsilon(t)$ | $\frac{1}{s}$ | $\mathrm{Re}\{s\} > 0$ |
| $e^{-at}\varepsilon(t)$ | $\frac{1}{s+a}$ | $\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$ |
| $-e^{-at}\varepsilon(-t)$ | $\frac{1}{s+a}$ | $\mathrm{Re}\{s\} < \mathrm{Re}\{-a\}$ |
| $t\varepsilon(t)$ | $\frac{1}{s^2}$ | $\mathrm{Re}\{s\} > 0$ |
| $t^n\varepsilon(t)$ | $\frac{n!}{s^{n+1}}$ | $\mathrm{Re}\{s\} > 0$ |
| $te^{-at}\varepsilon(t)$ | $\frac{1}{(s+a)^2}$ | $\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$ |
| $t^n e^{-at}\varepsilon(t)$ | $\frac{n!}{(s+a)^{n+1}}$ | $\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$ |
| $\sin(\omega_0 t)\varepsilon(t)$ | $\frac{\omega_0}{s^2+\omega_0^2}$ | $\mathrm{Re}\{s\} > 0$ |
| $\cos(\omega_0 t)\varepsilon(t)$ | $\frac{s}{s^2+\omega_0^2}$ | $\mathrm{Re}\{s\} > 0$ |
| $e^{-at}\cos(\omega_0 t)\varepsilon(t)$ | $\frac{s+a}{(s+a)^2+\omega_0^2}$ | $\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$ |
| $e^{-at}\sin(\omega_0 t)\varepsilon(t)$ | $\frac{\omega_0}{(s+a)^2+\omega_0^2}$ | $\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$ |
| $t\cos(\omega_0 t)\varepsilon(t)$ | $\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$ | $\mathrm{Re}\{s\} > 0$ |
| $t\sin(\omega_0 t)\varepsilon(t)$ | $\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$ | $\mathrm{Re}\{s\} > 0$ |

## Appendix B.2 Properties of the Bilateral Laplace Transform

| $x(t)$ | $X(s)=\mathcal{L}\{x(t)\}$ | ROC |
|---|---|---|
| Linearity $Ax_1(t)+Bx_2(t)$ | $AX_1(s)+BX_2(s)$ | ROC $\supseteq$ ROC$\{X_1\}$ $\cap$ROC$\{X_2\}$ |
| Delay $x(t-\tau)$ | $e^{-s\tau}X(s)$ | not affected |
| Modulation $e^{at}x(t)$ | $X(s-a)$ | Re$\{a\}$ shifted by Re$\{a\}$ to the right |
| 'Multiplication by $t$', Differentiation in the frequency domain $tx(t)$ | $-\frac{d}{ds}X(s)$ | not affected |
| Differentiation in the time domain $\frac{d}{dt}x(t)$ | $sX(s)$ | ROC $\supseteq$ ROC$\{X\}$ |
| Integration $\int_{-\infty}^{t} x(\tau)d\tau$ | $\frac{1}{s}X(s)$ | ROC $\supseteq$ ROC$\{X\}$ $\cap\{s : \mathrm{Re}\{s\} > 0\}$ |
| Scaling $x(at)$ | $\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$ | ROC scaled by a factor of $a$ |

## Appendix B.3 Fourier Transform Pairs

| $x(t)$ | $X(j\omega)=\mathcal{F}\{x(t)\}$ |
|---|---|
| $\delta(t)$ | $1$ |
| $1$ | $2\pi\delta(\omega)$ |
| $\dot{\delta}(t)$ | $j\omega$ |
| $\frac{1}{T}ш\left(\frac{t}{T}\right)$ | $ш\left(\frac{\omega T}{2\pi}\right)$ |
| $\varepsilon(t)$ | $\pi\delta(\omega)+\frac{1}{j\omega}$ |
| $\mathrm{rect}(at)$ | $\frac{1}{\lvert a\rvert}\mathrm{si}\left(\frac{\omega}{2a}\right)$ |
| $\mathrm{si}(at)$ | $\frac{\pi}{\lvert a\rvert}\mathrm{rect}\left(\frac{\omega}{2a}\right)$ |
| $\frac{1}{t}$ | $-j\pi\,\mathrm{sign}(\omega)$ |
| $\mathrm{sign}(t)$ | $\frac{2}{j\omega}$ |
| $e^{j\omega_0 t}$ | $2\pi\delta(\omega-\omega_0)$ |
| $\cos(\omega_0 t)$ | $\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$ |
| $\sin(\omega_0 t)$ | $j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$ |
| $e^{-\alpha\lvert t\rvert},\ \alpha>0$ | $\frac{2\alpha}{\alpha^2+\omega^2}$ |
| $e^{-a^2t^2}$ | $\frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}$ |

## Appendix B.4 Properties of the Fourier Transform

| | $x(t)$ | $X(j\omega)=\mathcal{F}\{x(t)\}$ |
|---|---|---|
| Linearity | $Ax_1(t)+Bx_2(t)$ | $AX_1(j\omega)+BX_2(j\omega)$ |
| Delay | $x(t-\tau)$ | $e^{-j\omega\tau}X(j\omega)$ |
| Modulation | $e^{j\omega_0 t}x(t)$ | $X(j(\omega-\omega_0))$ |
| 'Multiplication by $t$' Differentiation in the frequency domain | $tx(t)$ | $-\frac{dX(j\omega)}{d(j\omega)}$ |
| Differentiation in the time domain | $\frac{dx(t)}{dt}$ | $j\omega X(j\omega)$ |
| Integration | $\int_{-\infty}^{t} x(\tau)d\tau$ | $X(j\omega)\left[\pi\delta(\omega)+\frac{1}{j\omega}\right]$ $=\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$ |
| Scaling | $x(at)$ | $\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right),\quad a\in\mathbb{R}\backslash\{0\}$ |
| Convolution | $x_1(t)*x_2(t)$ | $X_1(j\omega)\cdot X_2(j\omega)$ |
| Multiplication | $x_1(t)\cdot x_2(t)$ | $\frac{1}{2\pi}X_1(j\omega)*X_2(j\omega)$ |
| Duality | $x_1(t)$ <br> $x_2(jt)$ | $x_2(j\omega)$ <br> $2\pi x_1(-\omega)$ |
| Symmetry relations | $x(-t)$ <br> $x^*(t)$ <br> $x^*(-t)$ | $X(-j\omega)$ <br> $X^*(-j\omega)$ <br> $X^*(j\omega)$ |
| Parseval theorem | $\int_{-\infty}^{\infty}\lvert x(t)\rvert^2\,dt$ | $\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2 d\omega$ |

**Appendix B.6 Properties of the $z$-Transform**

| Property | $x[k]$ | $X(z)$ | ROC |
|---|---|---|---|
| Linearity | $ax_1[k]+bx_2[k]$ | $aX_1(z)+bX_2(z)$ | ROC $\supseteq$ ROC$\{X_1\}\cap$ROC$\{X_2\}$ |
| Delay | $x[k-\kappa]$ | $z^{-\kappa}X(z)$ | ROC$\{x\}$; separate consideration of $z=0$ and $z\to\infty$ |
| Modulation | $a^k x[k]$ | $X\left(\frac{z}{a}\right)$ | ROC$=\left\{z \,\middle\vert\, \frac{z}{a} \in \text{ROC}\{x\}\right\}$ |
| Multiplication by $k$ | $kx[k]$ | $-z\frac{dX(z)}{dz}$ | ROC$\{x\}$; separate consideration of $z=0$ |
| Time inversion | $x[-k]$ | $X(z^{-1})$ | ROC$=\{z \,\vert\, z^{-1}\in \text{ROC}\{x\}\}$ |
| Convolution | $x_1[k]*x_2[k]$ | $X_1(z)\cdot X_2(z)$ | ROC $\supseteq$ ROC$\{x_1\}\cap$ROC$\{x_2\}$ |
| Multiplication | $x_1[k]\cdot x_2[k]$ | $\frac{1}{2\pi j}\oint X_1(\zeta)X_2\left(\frac{z}{\zeta}\right)\frac{1}{\zeta}d\zeta$ | multiply the limits of the ROC |

**Appendix B.5 Two-sided $z$-Transform Pairs**

| $x[k]$ | $X(z)=\mathcal{Z}\{x[k]\}$ | ROC |
|---|---|---|
| $\delta[k]$ | $1$ | $z\in\mathbb{C}$ |
| $\varepsilon[k]$ | $\frac{z}{z-1}$ | $\lvert z\rvert>1$ |
| $a^k\varepsilon[k]$ | $\frac{z}{z-a}$ | $\lvert z\rvert>\lvert a\rvert$ |
| $-a^k\varepsilon[-k-1]$ | $\frac{z}{z-a}$ | $\lvert z\rvert<\lvert a\rvert$ |
| $k\varepsilon[k]$ | $\frac{z}{(z-1)^2}$ | $\lvert z\rvert>1$ |
| $ka^k\varepsilon[k]$ | $\frac{az}{(z-a)^2}$ | $\lvert z\rvert>\lvert a\rvert$ |
| $\sin(\Omega_0 k)\varepsilon[k]$ | $\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$ | $\lvert z\rvert>1$ |
| $\cos(\Omega_0 k)\varepsilon[k]$ | $\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$ | $\lvert z\rvert>1$ |