NTNU – Trondheim
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Examination paper for TFY4280 Signal Processing

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Examination time (from-to): 0900 - 1300

Permitted examination support material:

- Simple calculator (according to NTNU exam regulations)
- K. Rottmann: Matematisk formelsamling
- Barnett and Cronin: Mathematical formulae
- Carl Angell og Bjørn Ebbe Lian: Fysiske størrelser og enheter, navn og symboler

Other information about the examp paper:

- Language: English
- Number of pages (including this page and attachments): 12
- Answer must be written in English or Norwegian. Number of points given to each sub-question is given in bold font. The maximum score for the exam is **100p**.
- **Attachment:** 2 pages with transform tables and properties.

Checked by:

Date: **Signature:**

Q1 (30p)

A) (10p) Describe the concept of *transfer function* for continuous-time and discrete-time LTI systems. What properties of the LTI system allows you to use transfer function to determine output signal for a given input signal. Explain.

For LTI system the transfer function connects the input and the output in the frequency domain (in the s, ω or z domains).

$$H(s/\omega/z) = \frac{Y(s/\omega/z)}{X(s/\omega/z)}$$

This description is valid for LTI systems and is connected to the fact that for LTI system, the output is connected to the input through the convolution with the impulse response function in the time domain. Transfer function does not contain information about the initial state of the system, so this has to be zero for this approach to provide the correct description of the output signal.

B) (10p) Find the transfer functions and the corresponding system equation for continuous-time and discrete-time LTI systems described by the impulse response functions given below

$$h_1(t) = \varepsilon(t)e^{-at}\sin(\omega_0 t)$$
$$h_2[n] = \alpha\delta[n] + (1-\alpha)h[n-1]$$
$$h_3[n] = [\underline{10}\ 9\ 8\ 7]$$

$$h_1(t) = \varepsilon(t)e^{-at}\sin(\omega_0 t)$$

using table:

$$\begin{aligned}
\frac{Y(s)}{X(s)} = H_1(s) &= \frac{\omega_0}{(s+a)^2+\omega_0^2} = \frac{\omega_0}{s^2+2sa+a^2+\omega_0^2} \\
X(s)\omega_0 &= Y(s)[s^2+2as+(a^2+\omega^2)] \\
x(t)\omega_0 &= \ddot{y}(t)+2a\dot{y}(t)+(a^2+\omega^2)y(t)
\end{aligned}$$

For $h_2[n]$ we need to use the z-transform

$$\begin{aligned}
H_2(z) &= \alpha + (1-\alpha)H(z)z^{-1} \\
H_2(z)[1-(1-\alpha)z^{-1}] &= \alpha \\
H_2(z) = \frac{\alpha}{1-(1-\alpha)z^{-1}} &= \frac{z\alpha}{z-(1-\alpha)} \\
Y(z)(z-(1-\alpha)) &= \alpha X(z)z \\
Y(z)z - Y(z)(1-\alpha) &= \alpha X(z)z \\
Y(z) - Y(z)z^{-1}(1-\alpha) &= \alpha X(z) \\
y[n] - (1-\alpha)y[n-1] &= \alpha x[n]
\end{aligned}$$

$$\begin{aligned}
h_3[n] &= [\underline{10}\ 9\ 8\ 7] \\
H_3(z) = 10 + 9z^{-1} + 8z^{-2} + 7z^{-3} &= \frac{Y(z)}{X(z)} \\
Y(z) &= (10 + 9z^{-1} + 8z^{-2} + 7z^{-3})X(z) \\
y[n] &= 10x[n] + 9x[n-1] + 8x[n-2] + 7x[n-3]
\end{aligned}$$

C) **(10p)** How can one use transfer function to describe the frequency response of a LTI system. How are these two concepts connected? Explain how to calculate frequency responses for systems with impulse given above. Where necessary, use sampling time $t_s = \frac{2\pi}{100}$s.

For $H_1(s)$ one need to calculate this for $s = j\omega$ and find the amplitude and phase respons by writing this in terms of

$$H(j\omega) = |H(j\omega)|e^{j\varphi}$$

where the two terms on the right side correspond to the amplitude and phase response. For $h_2[n]$ and $h_3[n]$ replace z with $z = e^{j\omega} = e^{j\Omega t_s}$ where ω is the discrete frequency. Then calculate

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\varphi}$$

here $\omega = t_s\Omega$

Q2 (30p)

A) **(10p)** Find the unilateral ($n \geq 0$) z-transform of

$$x[n] = 5\cos(3n)$$

We can calculate the transform of $5\cos(3n)$ by first rewriting the cosine function,

$$\cos(\omega n) = \frac{1}{2}\left(e^{-j\omega n} + e^{j\omega n}\right)$$

which has the transform (use table or calculate),

$$Z\{\cos(\omega n)\} = \frac{1}{2}\left(\frac{z}{z - e^{-j\omega}} + \frac{z}{z - e^{j\omega}}\right) = \frac{1}{2}\frac{2z^2 - z\left(e^{-j\omega} + e^{-j\omega}\right)}{z^2 - z\left(e^{-j\omega} + e^{j\omega} - 1\right)}$$
$$= \frac{z^2 - z\cos(\omega)}{z^2 - 2z\cos(\omega) + 1}$$

and from this we get

$$Z\{5\cos(3n)\} = 5\frac{z^2 - z\cos(3)}{z^2 - 2z\cos(3) + 1}$$

Here one could also simply use the expression provided in Table B.5, writing the expression with correct coefficients/frequency.

B) (10p) Determine the convolution

$$y(t) = e^{-at}\varepsilon(t) * \varepsilon(t) \tag{3}$$

using the Fourier transform method.

Using transform method:

$$Y(j\omega) = \mathcal{F}\left\{e^{-at}\epsilon(t)\right\}\mathcal{F}\left\{\epsilon(t)\right\} = \frac{1}{a + j\omega}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] =$$
$$= \frac{\pi\delta(\omega)}{a} + \frac{1}{j\omega(a + j\omega)}$$
$$\frac{1}{j\omega(a + j\omega)} = \frac{A}{j\omega} + \frac{B}{(a + j\omega)} = \frac{A(a + j\omega) + Bj\omega}{j\omega(a + j\omega)}$$
$$A + B = 0 \quad Aa = 1$$
$$\frac{1}{j\omega(a + j\omega)} = \frac{1}{aj\omega} - \frac{1}{a(a + j\omega)}$$
$$Y(j\omega) = \frac{1}{a}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] - \frac{1}{a}\frac{1}{(a + j\omega)}$$
$$y(t) = \frac{1}{a}\epsilon(t) - \frac{1}{a}e^{-at}\epsilon(t)$$

C) (10p) A system with a transfer function

$$H(s) = \frac{s-1}{s^2+3s+2} \tag{4}$$

is excited by white noise with power density N_0 giving an output signal $y(t)$. Determine the ACF $\varphi_{yy}(\tau)$, the mean μ_x and the variance σ_x^2 of the output signal $y(t)$

ACF: We need to find the magnitude of the frequency response:

$$H(s) = \frac{s-1}{s^2+3s+2}$$

$$\begin{aligned}|H(j\omega)|^2 = H(j\omega)H(-j\omega) = \frac{(j\omega-1)(-j\omega-1)}{[(2-\omega^2)+3j\omega]\,[(2-\omega^2)-3j\omega]} &= \\ = \frac{1+\omega^2}{(2-\omega^2)^2+9\omega^2} &= \\ = \frac{1+\omega^2}{(1+\omega^2)(4+\omega^2)} = \frac{1}{(4+\omega^2)}\end{aligned}$$

$$\Phi_{yy} = |H(j\omega)|^2\Phi_{xx} = N_0\frac{1}{(4+\omega^2)}$$

$$\varphi_{yy}(\tau) = \mathscr{F}^{-1}\{\Phi_{yy}(\tau)\}$$

Using the fact that

$$\mathscr{F}\{e^{-\alpha|t|}\} = \frac{2\alpha}{\alpha^2+\omega^2}$$

we get

$$\varphi_{yy}(\tau) = \frac{N_0}{4}e^{-2|\tau|}$$

Mean

$$\mu_y = \varphi_{yy}(\tau \to \infty) = 0$$

Variance:

$$\sigma_y^2 = \varphi_{yy}(0) - \mu_y^2 = N_0/4$$

Q3 (20p)

A) (10p) Show that:

$$\mathscr{L}\{t \cdot x(t)\} = -\frac{\mathrm{d}X(s)}{\mathrm{d}s}$$

B) (10p) Use above property to calculate output of LTI system where input $x(t) = te^{-9t}$ defined for $t > 0$ and the impulse response is given by:

$$H(s) = \frac{1}{(s+10)}$$

ANSWER:

Using the definition of the Laplace transform:

$$\mathscr{L}\{tx(t)\} = \int_{-\infty}^{\infty} tx(t)e^{-st}\mathrm{d}t = -\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}s}\left[x(t)e^{-st}\right]\mathrm{d}t =$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\int_{-\infty}^{\infty}\left[x(t)e^{-st}\right]\mathrm{d}t = -\frac{\mathrm{d}X(s)}{\mathrm{d}s}$$

Now we can get the Laplace transform of the input signal:

$$\mathscr{L}\left\{t \cdot e^{-at}\right\} = -\frac{\mathrm{d}\mathscr{L}\{e^{-at}\}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s+a}\right) = (s+a)^{-2}$$

$$Y(s) = \frac{1}{(s+10)} \cdot \frac{1}{(s+9)^2} = \left[\frac{A}{(s+10)} + \frac{B}{(s+9)^2} + \frac{C}{(s+9)}\right]$$

Need to solve by first letting $s = -10$ and $s = -9$:

$$\frac{1}{(9-10)^2} = A$$

$$A = 1$$

$$B = \frac{1}{-9+10} = 1/2$$

$$\frac{1}{s+10} + \frac{1}{(s+9)^2} + \frac{C}{(s+9)} =$$

$$= \frac{1}{(s+10)} \cdot \frac{1}{(s+9)^2}$$

$$\frac{(s+9)^2 + (s+10) + C(s+10)(s+9)}{(s+10)(s+9)^2} = \frac{1}{(s+10)(s+9)^2}$$

$$Cs^2 + s^2 = 0$$

$$C = -1$$

Check:

$$\frac{(s+9)^2+(s+10)-(s+10)(s+9)}{(s+10)(s+9)^2} =$$

$$= \frac{(s^2+18s+81+s+10-s^2-9s-10s-90)}{(s+10)(s+9)^2} =$$

$$= \frac{1}{(s+10)(s+9)^2} = OK$$

So:

$$Y(s) = \left[\frac{1}{(s+10)} + \frac{1}{(s+9)^2} - \frac{1}{(s+9)}\right]$$
$$y(t) = e^{-10t}\varepsilon(t) + te^{-9t}\varepsilon(t) - e^{-9t}\varepsilon(t)$$

Q4 (20p)

A) (10p) Explain the concept of discrete frequency by considerinbg Fourier transform of a sampled signal:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$

B) (10p) Describe the difference between DTFT and DFT. Show how to calculate both transforms for the signal defined by

$$x[n] = \begin{cases} 1 & 0 \le n < 10 \\ 0 & \text{otherwise} \end{cases}$$

NOTE: Here it is sufficient to express the transforms in terms of a power series.

If one calculate the fourier transform of a sampled signal $x_s(t)$ defined by a continuous time signal multiplied by a train of delta impulses, one will arrives at:

$$\begin{aligned} x_s(t) &= \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s) \\ \mathcal{F}\{x_s(t)\} &= \sum_{n=-\infty}^{\infty} x(nT_s)\mathcal{F}\{\delta(t-nT_s)\} = \sum_{n=-\infty}^{\infty} x(nT_s)\mathrm{e}^{-jn\Omega T_s} \end{aligned}$$

Where Ω is the frequency space for time contentious signal. If we now let $\omega = \Omega T_s$, then

$$X_s\left(e^{j\omega}\right) = X_s\left(e^{j\Omega T_s}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

were ω is the discrete frequency, connected to the real frequency by T_S. For given signal, DTFT:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1-e^{-j\omega M}}{1-e^{j\omega}} = \frac{1-e^{-j(\Omega T_s)M}}{1-e^{j\Omega T_s}}$$

For DFT, one would need to know the length of the signal in the discrete-time domain. Setting this to N, now discrete frequency is also discrete and

$$\omega = 2\pi k/N$$

$$\Omega = 2\pi k\frac{1}{NT_s}$$

Then $X(e^{j\omega})$ becomes a discrete function $X[k]$, where N is the length of the signal in the time domain, but also number of points in the fraquency space:

$$X[k] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\frac{2\pi k}{N}n} \quad 0 \leq k < N$$

Appendix B.1 Bilateral Laplace Transform Pairs

$x(t)$	$X(s) = \mathcal{L}\{x(t)\}$	ROC
$\delta(t)$	1	$s \in \mathbb{C}$
$\varepsilon(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
$e^{-at}\varepsilon(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$-e^{-at}\varepsilon(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} < \mathrm{Re}\{-a\}$
$t\varepsilon(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$
$t^n\varepsilon(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$
$te^{-at}\varepsilon(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$t^n e^{-at}\varepsilon(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
$\cos(\omega_0 t)\varepsilon(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\varepsilon(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$e^{-at}\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$t\cos(\omega_0 t)\varepsilon(t)$	$\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\} > 0$
$t\sin(\omega_0 t)\varepsilon(t)$	$\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\} > 0$

Appendix B.2 Properties of the Bilateral Laplace Transform

	$x(t)$	$X(s) = \mathcal{L}\{x(t)\}$	ROC
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(s) + BX_2(s)$	ROC $\supseteq$ ROC$\{X_1\}$ $\cap$ROC$\{X_2\}$
Delay	$x(t-\tau)$	$e^{-s\tau}X(s)$	not affected
Modulation	$e^{at}x(t)$	$X(s-a)$	$\mathrm{Re}\{a\}$ shifted by $\mathrm{Re}\{a\}$ to the right
'Multiplication by t', Differentiation in the frequency domain	$tx(t)$	$-\frac{d}{ds}X(s)$	not affected
Differentiation in the time domain	$\frac{d}{dt}x(t)$	$sX(s)$	ROC $\supseteq$ ROC$\{X\}$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	ROC $\supseteq$ ROC$\{X\}$ $\cap\{s : \mathrm{Re}\{s\} > 0\}$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	ROC scaled by a factor of a

Appendix B.3 Fourier Transform Pairs

$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\dot{\delta}(t)$	$j\omega$
$\frac{1}{T}\sqcup\!\sqcup\!\sqcup\left(\frac{t}{T}\right)$	$\sqcup\!\sqcup\!\sqcup\left(\frac{\omega T}{2\pi}\right)$
$\varepsilon(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\mathrm{rect}(at)$	$\frac{1}{\lvert a\rvert}\mathrm{si}\left(\frac{\omega}{2a}\right)$
$\mathrm{si}(at)$	$\frac{\pi}{\lvert a\rvert}\mathrm{rect}\left(\frac{\omega}{2a}\right)$
$\frac{1}{t}$	$-j\pi\,\mathrm{sign}(\omega)$
$\mathrm{sign}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$
$\sin(\omega_0 t)$	$j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$
$e^{-\alpha\lvert t\rvert},\ \alpha > 0$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$e^{-a^2t^2}$	$\frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}$

Appendix B.4 Properties of the Fourier Transform

	$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(j\omega) + BX_2(j\omega)$
Delay	$x(t-\tau)$	$e^{-j\omega\tau}X(j\omega)$
Modulation	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
'Multiplication by t' Differentiation in the frequency domain	$tx(t)$	$-\frac{dX(j\omega)}{d(j\omega)}$
Differentiation in the time domain	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$X(j\omega)\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$ $= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right),\quad a \in \mathbb{R}\backslash\{0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega)\cdot X_2(j\omega)$
Multiplication	$x_1(t)\cdot x_2(t)$	$\frac{1}{2\pi}X_1(j\omega) * X_2(j\omega)$
Duality	$x_1(t)$ $x_2(jt)$	$x_2(j\omega)$ $2\pi x_1(-\omega)$
Symmetry relations	$x(-t)$ $x^*(t)$ $x^*(-t)$	$X(-j\omega)$ $X^*(-j\omega)$ $X^*(j\omega)$
Parseval theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$

Appendix B.6 Properties of the z-Transform

Property	$x[k]$	$X(z)$	ROC
Linearity	$ax_1[k]+bx_2[k]$	$aX_1(z)+bX_2(z)$	ROC $\supseteq$ ROC$\{X_1\}\cap$ROC$\{X_2\}$
Delay	$x[k-\kappa]$	$z^{-\kappa}X(z)$	ROC$\{x\}$; separate consideration of $z=0$ and $z\to\infty$
Modulation	$a^k x[k]$	$X\left(\frac{z}{a}\right)$	ROC$=\left\{z\,\middle\vert\,\frac{z}{a}\in\text{ROC}\{x\}\right\}$
Multiplication by k	$kx[k]$	$-z\frac{dX(z)}{dz}$	ROC$\{x\}$; separate consideration of $z=0$
Time inversion	$x[-k]$	$X(z^{-1})$	ROC$=\{z\,\vert z^{-1}\in\text{ROC}\{x\}\}$
Convolution	$x_1[k]*x_2[k]$	$X_1(z)\cdot X_2(z)$	ROC $\supseteq$ ROC$\{x_1\}\cap$ROC$\{x_2\}$
Multiplication	$x_1[k]\cdot x_2[k]$	$\frac{1}{2\pi j}\oint X_1(\zeta)X_2\left(\frac{z}{\zeta}\right)\frac{1}{\zeta}d\zeta$	multiply the limits of the ROC

Appendix B.5 Two-sided z-Transform Pairs

$x[k]$	$X(z)=\mathcal{Z}\{x[k]\}$	ROC
$\delta[k]$	1	$z\in\mathbb{C}$
$\varepsilon[k]$	$\frac{z}{z-1}$	$\vert z\vert>1$
$a^k\varepsilon[k]$	$\frac{z}{z-a}$	$\vert z\vert>\vert a\vert$
$-a^k\varepsilon[-k-1]$	$\frac{z}{z-a}$	$\vert z\vert<\vert a\vert$
$k\varepsilon[k]$	$\frac{z}{(z-1)^2}$	$\vert z\vert>1$
$ka^k\varepsilon[k]$	$\frac{az}{(z-a)^2}$	$\vert z\vert>\vert a\vert$
$\sin(\Omega_0 k)\varepsilon[k]$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$	$\vert z\vert>1$
$\cos(\Omega_0 k)\varepsilon[k]$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$	$\vert z\vert>1$