NTNU – Trondheim
Norwegian University of
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Examination paper for TFY4280 Signal Processing

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Examination date: 27.05.2016

Examination time (from-to): 0900 - 1300

Permitted examination support material:

- Simple calculator (according to NTNU exam regulations)
- K. Rottmann: Matematisk formelsamling
- Barnett and Cronin: Mathematical formulae
- Carl Angell og Bjørn Ebbe Lian: Fysiske størrelser og enheter, navn og symboler

Other information about the examp paper:

- Language: English
- Number of pages (including this page and attachments): 9
- Answer must be written in English or Norwegian. Number of points given to each sub-question is given in bold font. The maximum score for the exam is **100p**.
- **Attachment:** 2 pages with transform tables and properties.

Checked by:

Date: **Signature:**

Q1. (30p) System $S_1\{\ \}$ is described by a transfer function $H_1(s)$ given below.

$$H_1(s) = \frac{1}{s^2 + 3s + 2}$$

A. Calculate unit step response for this system

$$\begin{aligned}
\frac{Y(s)}{X(s)} &= \frac{1}{s^2+3s+2} \\
Y(s) &= X(s)\frac{1}{s^2+3s+2} = \frac{1}{s}\frac{1}{s^2+3s+2} = \frac{1}{s(s+2)(s+1)} \\
&= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1} = \frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{(s+1)} = \\
&= \frac{2(s^2+3s+2) + 2s(s+1) - 4s(s+2)}{4s(s+2)(s+1)} = \\
&= \frac{4}{4s(s+2)(s+1)} = \frac{1}{s(s+2)(s+1)} \quad \text{ok!}
\end{aligned}$$

then

$$\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left\{\frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{(s-1)}\right\} = \\
&= \frac{1}{2}\varepsilon(t) + \frac{1}{2}e^{-2t}\varepsilon(t) - e^{-t}\varepsilon(t)
\end{aligned}$$

B. Design a discrete-time system which is equivalent to the system $S_1\{\ \}$ studied above. Find discrete time transfer function $H(z)$ and again calculate output if a discrete time unit step function $u[n]$ is given as an input. If necessary use $t_s = 1\text{s}$ and calculate only the few first terms of the output signal ($0 \le n < 3$).

Hint: to save time, use a difference equation for the system and calculate the unit step response in the time domain.

We can do this by first finding the differencial equation describing this system:

$$\frac{Y(s)}{X(s)} = \frac{1}{s^2+3s+2}$$
$$Y(s)(s^2+3s+2) = X(s)$$
$$s^2Y(s)+3sY(s)+2Y(s) = X(s)$$
$$\ddot{y}(t)+3\dot{y}(t)+2y(t) = x(t)$$
$$\dot{y}(t) = \frac{y[n]-y[n-1]}{t_s}$$

$$\ddot{y}(t) = \frac{\frac{y[n]-y[n-1]}{t_s} - \frac{y[n-1]-y[n-2]}{t_s}}{t_s} = \frac{1}{t_s^2}\left(y[n]-2y[n-1]+y[n-2]\right)$$

So the difference equation for this system will be:

$$\frac{1}{t_s^2}\left(y[n]-2y[n-1]+y[n-2]\right)+3\frac{y[n]-y[n-1]}{t_s}+2y[n] = x[n]$$
$$y[n]\left(\frac{1}{t_s^2}+\frac{3}{t_s}+2\right)-y[n-1]\left(\frac{2}{t_s^2}+\frac{3}{t_s}\right)+y[n-2]\frac{1}{t_s^2} = x[n]$$

To simplify we can evaluate expressions in the brackets

$$\left(\frac{1}{t_s^2}+\frac{3}{t_s}+2\right) = 1+3+2 = 6$$
$$\left(\frac{2}{t_s^2}+\frac{3}{t_s}\right) = 2+3 = 5$$
$$\frac{1}{t_s^2} = 1$$
$$6y[n]-5y[n-1]+y[n-2] = x[n]$$
$$y[n] = \frac{5}{6}y[n-1]-\frac{1}{6}y[n-2]+\frac{1}{6}x[n]$$

So, the transfer function can be obtained by taking the z-transform of this equation:

$$6y[n]-5y[n-1]+y[n-2] = x[n]$$
$$6Y(z)-5Y(z)z^{-1}+Y(z)z^{-2} = X(z)$$
$$\frac{Y(x)}{X(x)} = \frac{1}{6-5z^{-1}+z^{-2}} = \frac{z^2}{6z^2-5z+1}$$

To get the unit step response, we set:

$$\begin{aligned} X(z) &= \frac{z}{z-1} \\ Y(z) &= \frac{z^2}{6z^2-5z+1}\frac{z}{z-1} \\ \frac{Y(s)}{z} &= \frac{z^2}{6z^2-5z+1}\frac{1}{z-1} \\ &\ldots \end{aligned}$$

Unit step response in the time domain: we set $x[n] = 1$ for $n \geq 0$

$$\begin{aligned} y[0] &= \frac{1}{6} \\ y[1] &= \frac{5}{6}\frac{1}{6} + \frac{1}{6} = \frac{11}{36} \\ y[2] &= \frac{5}{6}\frac{11}{36} - \frac{1}{6}\frac{1}{6} + \frac{1}{6} = \frac{55-6+36}{216} = \frac{85}{216} \end{aligned}$$

Q2. (10p) Consider a discrete-time LTI system with a impulse response $h[n]$ given by:

$$h[n] = (-\alpha)^n u[n]$$

where $u[n]$ is the unit step function.

A. Is this system causal?

B. For what range of α-values is this system BIBO stable?

Since $h[n] = 0$ for $n < 0$, the system is causal. For BIBO stability, the impulse response function needs to be absolutelly integrable.

$$\sum_{n=-\infty}^{\infty} |-\alpha^n u[n]| = \sum_{n=0}^{\infty} |-\alpha^n| = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|} \qquad |\alpha| < 1$$

so the system will be BIBO stable for $|\alpha| < 1$.

Q3. (10p) Find the discrete-time Fourier transform (DTFT) of the rectangular pulse sequence given by

$$x[n] = u[n] - u[n-N]$$

where $u[n]$ is the unit step function. This discrete-time signal is sampled with a sampling frequency ω_s, write the expression for the transform both in the discrete frequency domain and in the frequency domain.

Note: it is enough to write the aswer as a fraction of two complex functions and you do not need to arrive at an elegant expression.

Using:

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

and the definition of DTFT

$$X(e^{j\omega}) = \mathcal{F}_* \{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \quad -\pi \leq \omega < \pi$$

we get

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega n}$$

$$X(e^{j\omega}) = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

where ω is the discrete frequency (not the same as the sampling frequency ω_s) and is related to frequency Ω by $\omega = \Omega t_s$. So, in the frequency domain, this will be

$$X(e^{j\omega}) = \frac{1-e^{-j\Omega t_s N}}{1-e^{-j\Omega t_s}}$$

and

$$t_s = \frac{2\pi}{\omega_s}$$

Q4. (10p) Find the inverse z-transform of

$$X(z) = z^2 \left(1 - \frac{1}{2} z^{-1}\right) \left(1 - z^{-1}\right) \left(1 + 2z^{-1}\right) \quad 0 < |z| < \infty$$

After multiplication we get:

$$X(z) = z^2 + \frac{1}{2} z - \frac{5}{2} + z^{-1}$$

Using the definition of the Z-transform we find:

$$x[n] = \left\{1, \frac{1}{2}, \underline{-\frac{5}{2}}, 1\right\}$$

(this can also be expressed as a series of discrete-time shifted $\delta[n]$-functions)

Q5. (10p) Find the z-transform of the following signal

$$x[n] = a^{-n}u[-n]$$

Uing the definition of the Z-transform we get:

$$\mathcal{Z}_b\{f[n]\} = \sum_{n=-\infty}^{\infty} f[n]\, z^{-n}$$

$$\mathcal{Z}_b\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=-\infty}^{0} a^{-n}z^{-n} = a^0z^0 + a^1z^1 + a^2z^2 + ... =$$

$$= \frac{1}{1-az} \qquad \text{ROC:}|za| < 1; \quad |z| < \frac{1}{|a|}$$

Q6. (20p) If the Fourier transform of a signal $x(t)$ is given by $X(\omega)$, find an expression for the Fourier transform $X_s(\omega)$ of signal $x_s(t)$ defined as:

$$x_s(t) = \delta_T(t)x(t)$$

where

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kt_s)$$

Explain how obtained expression relates to Nyquist sampling rate condition.

HINT: expression for the Fourier transform of a periodic function might be useful here:

$$F_p(\omega) = \sum_{n=-\infty}^{\infty} \omega_0 F(n\omega_0)\delta(\omega - n\omega_0)$$

We need to know FT of a train of delta impulses $\delta_T(t)$ (which is not the same as DTFT of this signal) and then use convolution in the frequency domain to find the tarnsform of a sampled function $x_s(t)$.

$$\mathcal{F}\left\{\sum_{k=-\infty}^{\infty} \delta(t - kt_s)\right\} = \sum_{n=-\infty}^{\infty} \omega_s \delta(\omega - n\omega_s)$$

and

$$\omega_s = \frac{2\pi}{t_s}$$

Now we can use convolution:

$$X_s(\omega) = \frac{1}{2\pi} X(\omega) * \sum_{n=-\infty}^{\infty} \omega_s \delta(\omega - n\omega_s) = \frac{\omega_s}{2\pi} \sum_{n=-\infty}^{\infty} F(\omega - n\omega_s)$$

Multiple copies of the transform will overlap if the sampleing frequency is less then $2\omega_{max}$.

Q7. **(10p)** What is defined by a power density spectrum of a random signal and how can it be calculated? Sketch power density spectrum of *white noise* and *band-limited white noise* signals.

If a random signal is stationary, power density spectrum can be calculated using the Fourier transform of the autocorrelation function, that is

$$\Phi_{xx}(j\omega) = \mathcal{F}\left\{\varphi_{xx}(\tau)\right\} \tag{1}$$

$$E\{|x(t)|^2\} = \varphi_{xx}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xx}(j\omega) e^{j\omega\tau} \mathrm{d}\omega \Bigg|_{\tau=0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xx}(j\omega) \mathrm{d}\omega$$

And therefore $\Phi_{xx}(j\omega)$ is the Power density spectrum. For signals for which Fourier transform exists, this can also be calculated by using the Parseval's theorem. For white noise signal (for which the autocorrelation function is a delta function), the power density is the same for all frequency ranges, so

$$\Phi_{xx}(j\omega) = N_0 \quad -\infty < \omega < \infty \tag{2}$$

for band-limited white noise,

$$\Phi_{xx}(j\omega) = N_0 \quad -\omega_{max} < \omega < \omega_{max} \tag{3}$$

Appendix B.1 Bilateral Laplace Transform Pairs

$x(t)$	$X(s)=\mathcal{L}\{x(t)\}$	ROC
$\delta(t)$	1	$s \in \mathbb{C}$
$\varepsilon(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\}>0$
$e^{-at}\varepsilon(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$-e^{-at}\varepsilon(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\}<\mathrm{Re}\{-a\}$
$t\varepsilon(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\}>0$
$t^n\varepsilon(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\}>0$
$te^{-at}\varepsilon(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$t^ne^{-at}\varepsilon(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$\sin(\omega_0t)\varepsilon(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\}>0$
$\cos(\omega_0t)\varepsilon(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\}>0$
$e^{-at}\cos(\omega_0t)\varepsilon(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$e^{-at}\sin(\omega_0t)\varepsilon(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$t\cos(\omega_0t)\varepsilon(t)$	$\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\}>0$
$t\sin(\omega_0t)\varepsilon(t)$	$\frac{2\omega_0s}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\}>0$

Appendix B.2 Properties of the Bilateral Laplace Transform

$x(t)$	$X(s)=\mathcal{L}\{x(t)\}$	ROC
Linearity $Ax_1(t)+Bx_2(t)$	$AX_1(s)+BX_2(s)$	ROC $\supseteq$ ROC$\{X_1\}$ $\cap$ROC$\{X_2\}$
Delay $x(t-\tau)$	$e^{-s\tau}X(s)$	not affected
Modulation $e^{at}x(t)$	$X(s-a)$	$\mathrm{Re}\{a\}$ shifted by $\mathrm{Re}\{a\}$ to the right
'Multiplication by t', Differentiation in the frequency domain $tx(t)$	$-\frac{d}{ds}X(s)$	not affected
Differentiation in the time domain $\frac{d}{dt}x(t)$	$sX(s)$	ROC $\supseteq$ ROC$\{X\}$
Integration $\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	ROC $\supseteq$ ROC$\{X\}$ $\cap\{s:\mathrm{Re}\{s\}>0\}$
Scaling $x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	ROC scaled by a factor of a

Appendix B.3 Fourier Transform Pairs

$x(t)$	$X(j\omega)=\mathcal{F}\{x(t)\}$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\dot{\delta}(t)$	$j\omega$
$\frac{1}{T}\text{Ш}\left(\frac{t}{T}\right)$	$\text{Ш}\left(\frac{\omega T}{2\pi}\right)$
$\varepsilon(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$\mathrm{rect}(at)$	$\frac{1}{\lvert a\rvert}\mathrm{si}\left(\frac{\omega}{2a}\right)$
$\mathrm{si}(at)$	$\frac{\pi}{\lvert a\rvert}\mathrm{rect}\left(\frac{\omega}{2a}\right)$
$\frac{1}{t}$	$-j\pi\mathrm{sign}(\omega)$
$\mathrm{sign}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0t)$	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$
$\sin(\omega_0t)$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$
$e^{-\alpha\lvert t\rvert},\ \alpha>0$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$e^{-a^2t^2}$	$\frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}$

Appendix B.4 Properties of the Fourier Transform

	$x(t)$	$X(j\omega)=\mathcal{F}\{x(t)\}$
Linearity	$Ax_1(t)+Bx_2(t)$	$AX_1(j\omega)+BX_2(j\omega)$
Delay	$x(t-\tau)$	$e^{-j\omega\tau}X(j\omega)$
Modulation	$e^{j\omega_0t}x(t)$	$X(j(\omega-\omega_0))$
'Multiplication by t' Differentiation in the frequency domain	$tx(t)$	$-\frac{dX(j\omega)}{d(j\omega)}$
Differentiation in the time domain	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$X(j\omega)\left[\pi\delta(\omega)+\frac{1}{j\omega}\right]$ $=\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right),\quad a\in\mathbb{R}\backslash\{0\}$
Convolution	$x_1(t)*x_2(t)$	$X_1(j\omega)\cdot X_2(j\omega)$
Multiplication	$x_1(t)\cdot x_2(t)$	$\frac{1}{2\pi}X_1(j\omega)*X_2(j\omega)$
Duality	$x_1(t)$ $x_2(jt)$	$x_2(j\omega)$ $2\pi x_1(-\omega)$
Symmetry relations	$x(-t)$ $x^*(t)$ $x^*(-t)$	$X(-j\omega)$ $X^*(-j\omega)$ $X^*(j\omega)$
Parseval theorem	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2\,dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2d\omega$

Appendix B.6 Properties of the z-Transform

Property	$x[k]$	$X(z)$	ROC
Linearity	$ax_1[k]+bx_2[k]$	$aX_1(z)+bX_2(z)$	ROC $\supseteq$ ROC$\{X_1\}\cap$ROC$\{X_2\}$
Delay	$x[k-\kappa]$	$z^{-\kappa}X(z)$	ROC$\{x\}$; separate consideration of $z=0$ and $z\to\infty$
Modulation	$a^k x[k]$	$X\left(\frac{z}{a}\right)$	ROC$=\left\{z \left\| \frac{z}{a} \in \text{ROC}\{x\}\right.\right\}$
Multiplication by k	$kx[k]$	$-z\frac{dX(z)}{dz}$	ROC$\{x\}$; separate consideration of $z=0$
Time inversion	$x[-k]$	$X(z^{-1})$	ROC$=\{z \,\|\, z^{-1}\in$ROC$\{x\}\}$
Convolution	$x_1[k]*x_2[k]$	$X_1(z)\cdot X_2(z)$	ROC $\supseteq$ ROC$\{x_1\}\cap$ROC$\{x_2\}$
Multiplication	$x_1[k]\cdot x_2[k]$	$\frac{1}{2\pi j}\oint X_1(\zeta)X_2\left(\frac{z}{\zeta}\right)\frac{1}{\zeta}d\zeta$	multiply the limits of the ROC

Appendix B.5 Two-sided z-Transform Pairs

$x[k]$	$X(z)=\mathcal{Z}\{x[k]\}$	ROC
$\delta[k]$	1	$z\in\mathbb{C}$
$\varepsilon[k]$	$\frac{z}{z-1}$	$\|z\|>1$
$a^k\varepsilon[k]$	$\frac{z}{z-a}$	$\|z\|>\|a\|$
$-a^k\varepsilon[-k-1]$	$\frac{z}{z-a}$	$\|z\|<\|a\|$
$k\varepsilon[k]$	$\frac{z}{(z-1)^2}$	$\|z\|>1$
$ka^k\varepsilon[k]$	$\frac{az}{(z-a)^2}$	$\|z\|>\|a\|$
$\sin(\Omega_0 k)\varepsilon[k]$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$	$\|z\|>1$
$\cos(\Omega_0 k)\varepsilon[k]$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$	$\|z\|>1$