NTNU – Trondheim
Norwegian University of
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Examination paper for TFY4280 Signal Processing

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Examination date: 02.06.2017

Examination time (from-to): 0900 - 1300

Permitted examination support material:

- Simple calculator (according to NTNU exam regulations)
- K. Rottmann: Matematisk formelsamling or an equilivalent, for example: Formulaires et tables: mathématiques, physique, chimie. Editions du Tricorne, Genève.
- Barnett and Cronin: Mathematical formulae
- Carl Angell og Bjørn Ebbe Lian: Fysiske størrelser og enheter, navn og symboler

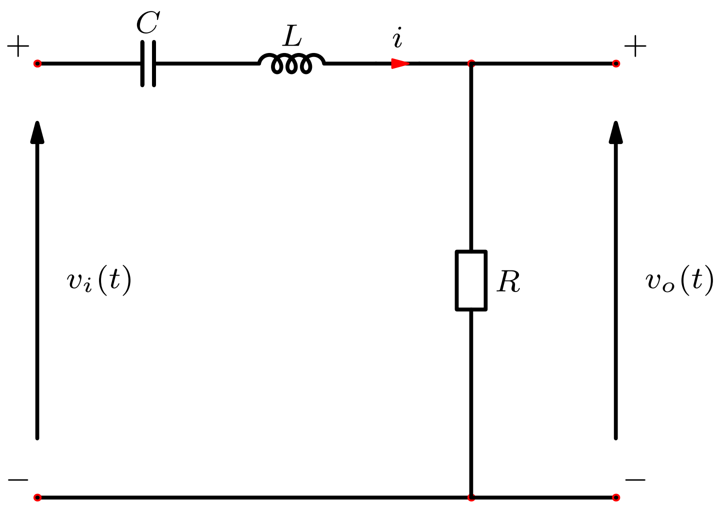
Other information about the examp paper:

- Language: English
- Number of pages (including this page and attachments): 10
- Answer must be written in English or Norwegian. Number of points given to each question is given in bold font. The maximum score for the exam is **100p**.
- **Attachment:** 2 pages with transform tables and properties.

Checked by:

Date: **Signature:**

Q1. **(35p)** Consider a simple circuit shown below, for which R = 10, C = 0.0235 and L = 10 in the appropriate s.i. units.



A. Write a differential equation which relates the output $v_o(t)$ with the input $v_i(t)$ for this system. **(5p)**

Answer:

$$Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{C}\int_0^t i(\tau)\mathrm{d}\tau = v_i(t)$$

$$v_o(t) + L/R\frac{\mathrm{d}v_o(t)}{\mathrm{d}t} + \frac{1}{RC}\int_0^t v_o(\tau)\mathrm{d}\tau = v_i(t)$$

B. Give a definitions of an impulse response function, a transfer function and a frequency response function. **(5p)**

Answer:

$$h(t) = S\{\delta(t)\}$$

$$H(s) = \frac{Y(s)}{X(s)}$$

$$H(j\omega) = H(s)|_{s=j\omega}$$

C. Calculate the impulse response function for the system given above. It is sufficient to express $h(t)$ as a sum of complex exponential functions. Explain how you would used this function to calculate the output signal $y(t)$ for a given input signal $x(t)$. **(10p)**

Answer:

$$Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{C}\int_0^t i(\tau)\mathrm{d}\tau = v_i(t)$$

$$v_o(t) + L/R\frac{\mathrm{d}v_o(t)}{\mathrm{d}t} + \frac{1}{RC}\int_0^t v_o(\tau)\mathrm{d}\tau = v_i(t)$$

$$V_o(s) + \frac{sL}{R}V_o(s) + \frac{1}{sRC}V_o(s) = V_i(s)$$

$$V_o\left(1 + \frac{sL}{R} + \frac{1}{sRC}\right) = V_i(s)$$

$$H(s) = \frac{1}{1 + \frac{sL}{R} + \frac{1}{sRC}} = \frac{s}{s + \frac{s^2L}{R} + \frac{1}{RC}} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

For given values of the components we get $R/L = 1$ and $1/CL = 4.25$

$$H(s) = \frac{s}{s^2 + s + 4.25}$$

Which has zeros at

$$p_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = -0.5 \pm \frac{\sqrt{1-17}}{2} = -0.5 \pm 2j$$

So we can write the transfer functions as:

$$\begin{aligned}
H(s) &= \frac{A}{s + 0.5 - 2j} + \frac{B}{s + 0.5 + 2j} = \\
&= \frac{s}{s^2 + s + 4.25} = \frac{s}{(s + 0.5 - 2j)(s + 0.5 + 2j)} \\
A &= \left.\frac{s}{(s + 0.5 + 2j)}\right|_{s=-0.5+2j} = \\
&= \frac{-0.5 + 2j}{-0.5 + 2j + 0.5 + 2j} = \frac{-0.5 + 2j}{4j} = 0.5 + 0.125j \\
B &= \left.\frac{s}{(s + 0.5 - 2j)}\right|_{s=-0.5-2j} = \\
&= \frac{-0.5 - 2j}{-0.5 - 2j + 0.5 - 2j} = \frac{-0.5 - 2j}{-4j} = 0.5 - 0.125j
\end{aligned}$$

Impulse response will be the inverse Laplace transform of this function:

$$H(s) = \frac{0.5 + 0.125j}{s + 0.5 - 2j} + \frac{0.5 - 125j}{s + 0.5 + 2j}$$
$$h(t) = \varepsilon(t)\left(Ae^{-(0.5-2j)t} + Be^{-(0.5+2j)t}\right)$$

D. Calculate the transfer function and the frequency response function of this system. For the frequency response function, you do not need to arrive at a elegant expressions. **(5p)**

Answer: For the frequency response, we set $s = j\omega$

$$H(j\omega) = \frac{j\omega}{j\omega^2 + j\omega + 4.25}$$
$$= \frac{0.5 + 0.125j}{j\omega + 0.5 - 2j} + \frac{0.5 - 125j}{j\omega + 0.5 + 2j}$$

E. Draw a zero/poles diagram. **(5p)**

Answer: The zeros/pole diagram includes zero at the origin and two poles at $s = -0.5 \pm 2j$

F. What kind of circuit is this? **(5p)**

Answer: Band pass filter. We can see it from the fact that $H(j\omega) = 0$ for $\omega = 0$ and $\omega = \infty$.

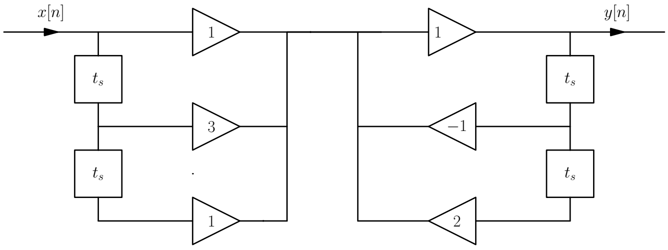
Q2. (10p) Find the z-transform of the following signal:

$$x[n] = \{0, 1, 2, 3, \underline{4}, 3, 2, 1, 0\}$$

Answer: Using definition of the z-transform we can write:

$$\begin{aligned} X(z) &= \mathcal{Z}_b\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \\ &= x[-3]\, z^3 + x[-2]\, z^2 + x[-1]\, z^1 + x[0]\, z^0 + x[1]\, z^{-1} + x[2]\, z^{-2} + x[3]\, z^{-3} = \\ &= 1z^3 + 2z^2 + 3z^1 + 4z^0 + 3z^{-1} + 2z^{-2} + 1z^{-3} \end{aligned}$$

Q3. (20p) Consider a discrete-time system described by a block diagram shown below.



A. Use this block diagram to calculate a discrete-time impulse response function for this system. Calculate for $n < 6$. **(10p)**

Answer: First we need a difference equation, then we can just set input as $\delta[0]$ and calculate the impulse response. From the diagram:

$$y[n] = x[n] + 3x[n-1] + x[n-2] - y[n-1] + 2y[n-2]$$

From that equation we get:

$$\begin{aligned}
h[0] &= \delta[0] + +3\delta[-1] + \delta[-2] - h[-1] + 2h[-2] = 1 \\
h[1] &= \delta[1] + 3\delta[0] + \delta[-1] - h[0] + 2h[-1] = 3 - 1 = 2 \\
h[2] &= \delta[2] + 3\delta[1] + \delta[0] - h[1] + 2h[0] = 1 - 2 + 2 = 1 \\
h[3] &= \delta[3] + 3\delta[2] + \delta[1] - h[2] + 2h[1] = -1 + 4 = 3 \\
h[4] &= \delta[4] + 3\delta[3] + \delta[2] - h[3] + 2h[2] = -3 + 2 = -1 \\
h[5] &= \delta[5] + 3\delta[4] + \delta[3] - h[4] + 2h[3] = 1 + 6 = 7
\end{aligned}$$

B. Use a method of choice to determine output signal $y[n]$ for $n < 8$ and the input signal given below. **(10p)**

$$x[n] = \{\underline{1}, 0, 1, 0\} \quad \text{and 0 otherwise.}$$

Q4. **(10p)** Determine the convolution

$$y(t) = e^{-at}\epsilon(t) * \epsilon(t)$$

using direct method (definition) and using Fourier transform method.

Answer: Using the definition we find for $t \geq 0$ (the convolution is zero for $t < 0$):

$$y(t) = \int_{-\infty}^{\infty} e^{-a(t-\beta)}\epsilon(t-\beta)\epsilon(\beta)\mathrm{d}\beta = e^{-at}\int_{-\infty}^{\infty} e^{a\beta}\epsilon(t-\beta)\epsilon(\beta)\mathrm{d}\beta = e^{-at}\int_{0}^{t} e^{a\beta}\mathrm{d}\beta =$$

$$= e^{-at}\,\frac{1}{a}e^{a\beta}\Big|_{\beta=0}^{\beta=t} = \frac{1}{a}(1 - e^{-at})$$

so

$$y(t) = \frac{1}{a}(1 - e^{-at})\epsilon(t)$$

Using transform method:

$$Y(j\omega) = \mathcal{F}\left\{e^{-at}\epsilon(t)\right\}\mathcal{F}\left\{\epsilon(t)\right\} = \frac{1}{a + j\omega}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] = \frac{\pi\delta(\omega)}{a} + \frac{1}{j\omega(a + j\omega)}$$

$$\frac{1}{j\omega(a + j\omega)} = \frac{A}{j\omega} + \frac{B}{(a + j\omega)} = \frac{A(a + j\omega) + Bj\omega}{j\omega(a + j\omega)}$$

$$A + B = 0 \quad Aa = 1$$

$$\frac{1}{j\omega(a + j\omega)} = \frac{1}{aj\omega} - \frac{1}{a(a + j\omega)}$$

$$Y(j\omega) = \frac{1}{a}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] - \frac{1}{a}\frac{1}{(a + j\omega)}$$

$$y(t) = \frac{1}{a}\epsilon(t) - \frac{1}{a}e^{-at}\epsilon(t)$$

Q5. (25p)

A. Use definition of of a unilateral z-transform ($\mathcal{Z}$) to determine

$$\mathcal{Z}\left\{u[n]e^{\beta n}\right\} =$$

and then use this to calculate $\mathcal{Z}\left\{u[n]cos(bn)\right\}$ **(10p)**

Answer:

$$X(z) = \mathcal{Z}\left\{x[n]\right\} = \sum_{n=0}^{\infty} x[n]\, z^{-n} = \sum_{n=0}^{\infty} e^{\beta n} z^{-n} =$$

$$= 1 + e^{1\beta}z^{-1} + e^{2\beta}z^{-2} + e^{3\beta}z^{-3} + ... = \frac{1}{1 - e^{\beta}z^{-1}} = \frac{z}{z - e^{\beta}}$$

With a region of convergence

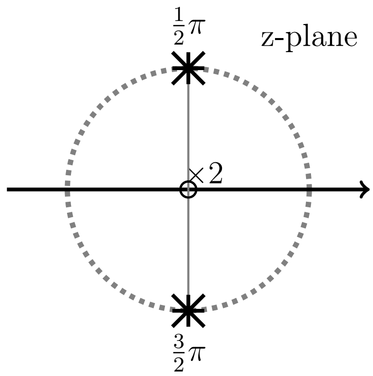
$$|e^{\beta}z^{-1} < 1|$$

$$\text{ROC:} \quad |e^{\beta}| < |z|$$

This transform has a pole at $z = e^{\beta}$ and a zero at $z = 0$.

$$\begin{aligned}
\cos bn &= \frac{1}{2}\left[e^{jbn}+e^{-jbn}\right] \\
\mathcal{Z}\left\{u[n]\cos bn\right\} &= \frac{1}{2}\mathcal{Z}\left\{e^{jbn}\right\}+\frac{1}{2}\mathcal{Z}\left\{e^{-jbn}\right\} = \\
&= \frac{1}{2}\left(\frac{z}{z-e^{jb}}\right)+\frac{1}{2}\left(\frac{z}{z-e^{-jb}}\right) = \\
&= \frac{z}{2}\left(\frac{z-e^{-jb}+z-e^{jb}}{(z-e^{jb})(z-e^{-jb})}\right) = \frac{z(z-\cos b)}{z^2-2z\cos b+1}
\end{aligned}$$

B. Zeros/poles diagram on the z-plane for a transfer function $H(z)$ of a LTI discrete-time system is given below. Use this diagram to find the difference equation which describes this system. **(15p)**



Answer: Signal described by this diagram as a transfer functions with poles at $z=\pm j=e^{\pm j\pi/2}$. Using the formula derived above we can write:

$$\begin{aligned}
H(z) &= \frac{z}{z-e^{j\pi/2}}+\frac{z}{z-e^{-j\pi/2}} = 2\frac{z(z-\cos(\pi/2))}{z^2-2z\cos(\pi/2)+1} = \\
&= 2\frac{z^2}{z^2+1}
\end{aligned}$$

Using the transfer function for a discrete-time LTI system:

$$\begin{aligned}
H(z) = \frac{Y(z)}{X(z)} &= 2\frac{z^2}{z^2+1} \\
Y(z)(z^2+1) &= X(z)[2z^2] \\
Y(z)(1+z^{-2}) &= 2X(z) \\
y[n]+y[n-2] &= 2x[n] \\
y[n] &= 2x[n]-y[n-2]
\end{aligned}$$

Appendix B.1 Bilateral Laplace Transform Pairs

$x(t)$	$X(s)=\mathcal{L}\{x(t)\}$	ROC
$\delta(t)$	1	$s \in \mathbb{C}$
$\varepsilon(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\}>0$
$e^{-at}\varepsilon(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$-e^{-at}\varepsilon(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\}<\mathrm{Re}\{-a\}$
$t\varepsilon(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\}>0$
$t^n\varepsilon(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\}>0$
$te^{-at}\varepsilon(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$t^ne^{-at}\varepsilon(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\}>0$
$\cos(\omega_0 t)\varepsilon(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\}>0$
$e^{-at}\cos(\omega_0 t)\varepsilon(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$e^{-at}\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$t\cos(\omega_0 t)\varepsilon(t)$	$\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\}>0$
$t\sin(\omega_0 t)\varepsilon(t)$	$\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\}>0$

Appendix B.2 Properties of the Bilateral Laplace Transform

$x(t)$	$X(s)=\mathcal{L}\{x(t)\}$	ROC
Linearity $Ax_1(t)+Bx_2(t)$	$AX_1(s)+BX_2(s)$	ROC $\supseteq$ ROC$\{X_1\}$ $\cap$ROC$\{X_2\}$
Delay $x(t-\tau)$	$e^{-s\tau}X(s)$	not affected
Modulation $e^{at}x(t)$	$X(s-a)$	$\mathrm{Re}\{a\}$ shifted by $\mathrm{Re}\{a\}$ to the right
'Multiplication by t', Differentiation in the frequency domain $tx(t)$	$-\frac{d}{ds}X(s)$	not affected
Differentiation in the time domain $\frac{d}{dt}x(t)$	$sX(s)$	ROC $\supseteq$ ROC$\{X\}$
Integration $\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	ROC $\supseteq$ ROC$\{X\}$ $\cap\{s:\mathrm{Re}\{s\}>0\}$
Scaling $x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	ROC scaled by a factor of a

Appendix B.3 Fourier Transform Pairs

$x(t)$	$X(j\omega)=\mathcal{F}\{x(t)\}$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\dot{\delta}(t)$	$j\omega$
$\frac{1}{T}⊥⊥⊥\left(\frac{t}{T}\right)$	$⊥⊥⊥\left(\frac{\omega T}{2\pi}\right)$
$\varepsilon(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$\mathrm{rect}(at)$	$\frac{1}{\lvert a\rvert}\mathrm{si}\left(\frac{\omega}{2a}\right)$
$\mathrm{si}(at)$	$\frac{\pi}{\lvert a\rvert}\mathrm{rect}\left(\frac{\omega}{2a}\right)$
$\frac{1}{t}$	$-j\pi\,\mathrm{sign}(\omega)$
$\mathrm{sign}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$
$\sin(\omega_0 t)$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$
$e^{-\alpha\lvert t\rvert},\ \alpha>0$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$e^{-a^2t^2}$	$\frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}$

Appendix B.4 Properties of the Fourier Transform

	$x(t)$	$X(j\omega)=\mathcal{F}\{x(t)\}$
Linearity	$Ax_1(t)+Bx_2(t)$	$AX_1(j\omega)+BX_2(j\omega)$
Delay	$x(t-\tau)$	$e^{-j\omega\tau}X(j\omega)$
Modulation	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
'Multiplication by t' Differentiation in the frequency domain	$tx(t)$	$-\frac{dX(j\omega)}{d(j\omega)}$
Differentiation in the time domain	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$X(j\omega)\left[\pi\delta(\omega)+\frac{1}{j\omega}\right]$ $=\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right),\quad a\in\mathbb{R}\backslash\{0\}$
Convolution	$x_1(t)*x_2(t)$	$X_1(j\omega)\cdot X_2(j\omega)$
Multiplication	$x_1(t)\cdot x_2(t)$	$\frac{1}{2\pi}X_1(j\omega)*X_2(j\omega)$
Duality	$x_1(t)$ $x_2(jt)$	$x_2(j\omega)$ $2\pi x_1(-\omega)$
Symmetry relations	$x(-t)$ $x^*(t)$ $x^*(-t)$	$X(-j\omega)$ $X^*(-j\omega)$ $X^*(j\omega)$
Parseval theorem	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2\,dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2 d\omega$

Appendix B.6 Properties of the z-Transform

Property	$x[k]$	$X(z)$	ROC
Linearity	$ax_1[k]+bx_2[k]$	$aX_1(z)+bX_2(z)$	ROC $\supseteq$ ROC$\{X_1\}\cap$ROC$\{X_2\}$
Delay	$x[k-\kappa]$	$z^{-\kappa}X(z)$	ROC$\{x\}$; separate consideration of $z=0$ and $z\to\infty$
Modulation	$a^k x[k]$	$X\left(\frac{z}{a}\right)$	ROC$=\left\{z \middle\| \frac{z}{a} \in \text{ROC}\{x\}\right\}$
Multiplication by k	$kx[k]$	$-z\frac{dX(z)}{dz}$	ROC$\{x\}$; separate consideration of $z=0$
Time inversion	$x[-k]$	$X(z^{-1})$	ROC$=\{z \mid z^{-1}\in$ROC$\{x\}\}$
Convolution	$x_1[k]*x_2[k]$	$X_1(z)\cdot X_2(z)$	ROC $\supseteq$ ROC$\{x_1\}\cap$ROC$\{x_2\}$
Multiplication	$x_1[k]\cdot x_2[k]$	$\frac{1}{2\pi j}\oint X_1(\zeta)X_2\left(\frac{z}{\zeta}\right)\frac{1}{\zeta}d\zeta$	multiply the limits of the ROC

Appendix B.5 Two-sided z-Transform Pairs

$x[k]$	$X(z)=\mathcal{Z}\{x[k]\}$	ROC
$\delta[k]$	1	$z\in\mathbb{C}$
$\varepsilon[k]$	$\frac{z}{z-1}$	$\|z\|>1$
$a^k\varepsilon[k]$	$\frac{z}{z-a}$	$\|z\|>\|a\|$
$-a^k\varepsilon[-k-1]$	$\frac{z}{z-a}$	$\|z\|<\|a\|$
$k\varepsilon[k]$	$\frac{z}{(z-1)^2}$	$\|z\|>1$
$ka^k\varepsilon[k]$	$\frac{az}{(z-a)^2}$	$\|z\|>\|a\|$
$\sin(\Omega_0 k)\varepsilon[k]$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$	$\|z\|>1$
$\cos(\Omega_0 k)\varepsilon[k]$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$	$\|z\|>1$