Q1 DTFT and DFT (25p)

1. Explain what is described by the term "discrete frequency".

   Answer: "discrete frequency" ω is a frequency space used to describe the frequency spectrum of a discrete signal. ω can be both continues (as for aperiodic discrete signals and DTFT) or discrete (as for periodic discrete signals and DFT). Discrete frequency is connected to real frequency by $\Omega = \omega t_s$, where t_s is the sampling time of the signal in the time domain.

2. Find DTFT of the following discrete time signal:

$$x_1[n] = a^n u[n]$$

   where $u[n]$ is the discrete unit step function. Use $t_s = 1$ms.

   Answer:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} = \sum_{n=0}^{\infty} a^n e^{-jn\omega} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n = \frac{1}{1 - ae^{-j\omega}}$$

   We can for example, use the Euler formula to find the amplitude and the phase:

$$\begin{aligned}
X(e^{j\omega}) &= \frac{1}{1 - ae^{-j\omega}} = \frac{1}{1 - a\cos(\omega) + aj\sin(\omega)} = \\
&= \frac{1 - a\cos(\omega) - aj\sin(\omega)}{(1 - a\cos(\omega))^2 - (ja\sin(\omega))^2} = \\
&= \frac{1 - a\cos(\omega) - aj\sin(\omega)}{1 - 2a\cos(\omega) + a^2\cos^2(\omega) + a^2\sin^2(\omega)} = \\
&= \frac{1 - a\cos(\omega) - aj\sin(\omega)}{1 - 2a\cos(\omega) + a^2} \\
\left|X(e^{j\omega})\right| &= \frac{1}{\sqrt{(1 - 2a\cos(\omega) + a^2)}}
\end{aligned}$$

3. Sketch the amplitude response for $a = 0.8$ using discrete frequency on the x-axis.

   Answer: See Figure 1.

4. Sketch the amplitude response for $a = 0.8$ using frequency on the x-axis.

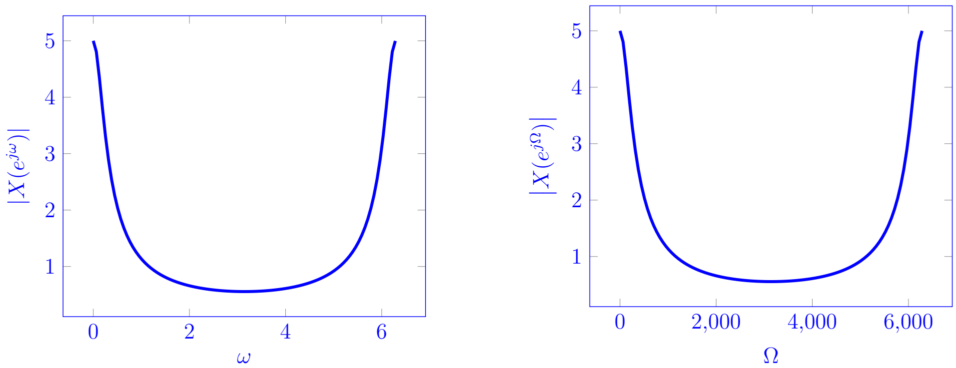


Figure 1: $X(e^{j\omega})$ plotted for discrete frequency ω between 0 and 2π and frequency Ω between 0 and Ω_s.

Answer:

$$\Omega = \omega t_s$$

so the frequency range should be $\Omega \in \left(0, \frac{2\pi}{t_s}\right)$, $\Omega \in (0, \Omega_s)$ or $\Omega \in (-\Omega_s/2, \Omega_s/2)$

5. What would you needed to be able to calculate DFT of the same signal? What would be the main difference between DFT and DTFT? Please explain.

   Answer: DFT is defined for signals which are discrete and periodic in the time domain. We therefore would need to define a periodic signal

$$\begin{aligned} x_2[n] &= a^n u[n] \quad 0 \le n < N \\ x_2[n+N] &= x_2[n] \end{aligned}$$

For such signal, DFT is defined as:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi nk/N} \quad 0 \le k \le N-1 \tag{1}$$

and the discrete frequency is now discrete and given by $\omega_k = 2\pi k/N$. DFT is discrete in the frequency domain, defined only for integer values of k and periodic with periodicity of 2π.

Q2 System output and frequency response. (25p)

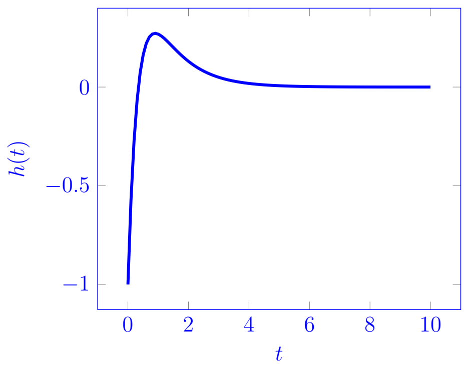
1. A LTI system is characterized by the impulse response function given below. Calculate and plot the response of that system to a unit step input $x_1(t) = \varepsilon(t)$ and to a delta impulse input $x_2(t) = \delta(t)$.

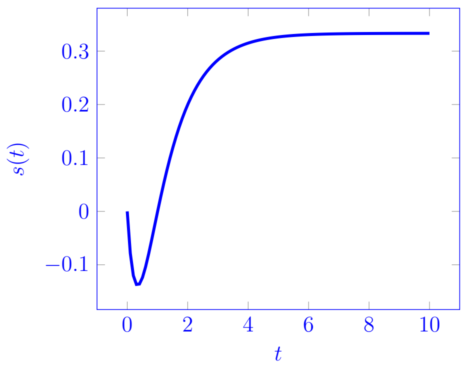
$$h(t) = \varepsilon(t)\mathrm{e}^{-t} - 2\mathrm{e}^{-3t}\varepsilon(t)$$

Also here, $\varepsilon(t)$ is the unit step function.

Answer:

$$\begin{aligned}
H(s) &= \frac{Y(s)}{X(s)} = \frac{1}{s+1} - \frac{2}{s+3} = \frac{s+3-2s-2}{(s+3)(s+1)} = \frac{1-s}{(s+3)(s+1)} \\
Y(s) &= X(s)H(s) \\
Y(s) &= \frac{1-s}{(s+3)(s+1)}\frac{1}{s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3} = \frac{1}{3s} - \frac{1}{s+1} + \frac{2}{3(s+3)} \\
y(t) &= \varepsilon(t)\left[\frac{1}{3} - \mathrm{e}^{-t} + \frac{2}{3}\mathrm{e}^{-3t}\right]
\end{aligned}$$





2. Calculate frequency response of this system for $\omega = 0$ and $\omega = 2\pi\mathrm{s}^{-1}$. Explain how you would calculate the frequency response for any given frequency.

Answer: Frequency response is given by

$$H(j\omega) = H(s)|_{s=j\omega}$$

Where $H(s)$ has been calculated above. To get an general expression for the frequency response, we need to substitute $s = j\omega$ and calculate $H(j\omega)$. For the two frequencies given in the problem text, one can use the simple expression for $H(s)$ obtained above.

$$H(j\omega) = \frac{1}{1+j\omega} - \frac{2}{3+j\omega}$$

For $\omega = 0$

$$|H(j\omega)| = 1 - \frac{2}{3} = \frac{1}{3}$$

For $\omega = 2\pi$

$$\begin{aligned} H(2\pi j) &= \frac{1}{1+2\pi j} - \frac{2}{3+2\pi j} = \frac{1-2\pi j}{1+4\pi^2} - \frac{6+4\pi j}{9+4\pi^2} = \\ &= 0.0247 - 0.155j - 0.1238 + 0.2592j = \\ &= -0.0991 + 0.1040j \end{aligned}$$

$|H(2\pi j)| = 0.1436$

Q3 Stochastic Signals (25p)

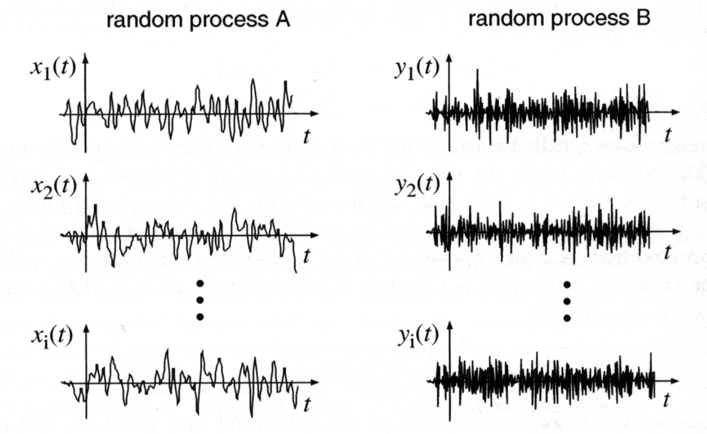
1. What is an "ergodic random process"? What is a "stationary random process"?

   Answer: A stationary random process for which the time-averages of each sample function are the same as the ensemble averages is called ergodic random process.

   A random process is stationary if its **2nd order expected values** only depend on the difference τ between t_1 and t_2, that is:

$$E\{f(x(t_1), x(t_2))\} = E\{f(x(t_1), x(t_1+\tau))\} \tag{2}$$

2. Define auto-correlation function (ACF). Sketch ACF for two signals, for which few sample functions are shown below, assuming that they are drawn on the same time scale.



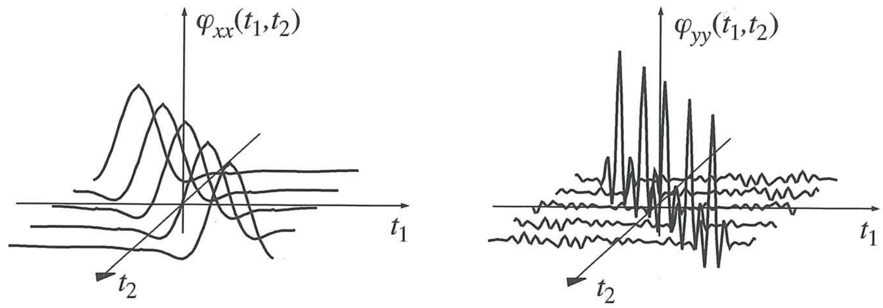
Answer:

$$\varphi_{xx}(t_1, t_2) = E\{x(t_1)x(t_2)\} \tag{3}$$

For a stationary process, this simplifies to

$$\varphi_{xx}(t_1, t_2) = \varphi_{xx}(\tau) = E\{x(t_1)x(t_1 + \tau)\} \tag{4}$$

In general ACF is a 2D function defined for any combination of t_1 and t_2. For stationary process, it becomes a 1D function of τ. Slower rate of change for the process A, would result in slower decay of the ACF, as illustrated on the figure below.



Q4 Filters. (25p)

1. For time-discrete systems, filters are often characterized as IIR or FIR. Explain what is described by these terms.

   Answer:

   IIR: infinite impulse response, system described by a recursive equation containing both $x[k]$ and $y[k]$ terms (input and output). $H(z)$ has both zeros and poles. $h[k]$ is given by a recursive equation.

$$\begin{aligned}\sum_{n=0}^{N} a_n y[k-n] &= \sum_{n=0}^{M} b_n x[k-n] \\ y[k] &= \sum_{n=0}^{M} b_n x[k-n] - \sum_{n=1}^{N} a_n y[k-n]\end{aligned}$$

   If we put $x[k] = \delta[k]$, then we can see that

$$h[k] = \sum_{n=0}^{M} b_n \delta[k-n] - \sum_{n=1}^{N} a_n h[k-n]$$

So, the impulse response function at time point k depends on both b_k coefficient (delta function in the first sum is not zero only for $k = n$) , but also on previous values of the impulse response function ($h[k-1]$, $h[k-2]$, etc)

FIR: finite impulse response.

$$y[k] \quad = \quad \sum_{n=0}^{M} b_n x[k-n]$$

If we put $x[k] = \delta[k]$, then we can see that

$$h[k] = \sum_{n=0}^{M} b_n x[k-n] = \{\underline{b_0}, b_1, b_2, ..., b_M\}$$

2. For which of these filter types, can we use discrete convolution to calculate output for an input signal similar to $x[n] = \{\underline{1}, 1, 1, 1, 1, 1\}$? Do not calculate but explain.

   Answer: Only for FIR, as in as to do an exact calculation we would need to include infinite number of terms for $h[n]$ in the case of IIR filter.

3. A low pass filters is described by the difference equation given below. Use zeros/poles diagram on an appropriate frequency plane to illustrate that this system is indeed a low pas filter.

   $$y[n] = \alpha x[n] + (1-\alpha) y[n-1]$$

   where α is a constants and $0 < \alpha < 1$

   Answer: To find zeros/poles of the transfer function we need to take z-transform of the difference equation and calculate $H(z)$.

   $$H(z) = \frac{\alpha z}{z - (1-\alpha)}$$

   This transform has a pole at $z = 1 - \alpha$ which is located on the horizontal axis, on the right hand side of the z-plane and for α approaching zero, the pole will be located close to the DC frequency, resulting in a large amplitude of the transfer function for these frequencies.

4. Is this a IIR or FIR filter? Please explain.

Answer: It is an IIR filter. One can see it for example by calculating few terms of the impulse response function.

$$\begin{aligned} h[0] &= \alpha \\ h[1] &= \alpha(1-\alpha) \\ h[2] &= \alpha(1-\alpha)^2 \\ h[2] &= \alpha(1-\alpha)^3 \\ &\dots \end{aligned}$$

One could transfer this filter to FIR by setting

$$h[n] = 0 \quad \text{for} \quad n > N$$

Where $N > 0$.

Appendix B.1 Bilateral Laplace Transform Pairs

$x(t)$	$X(s)=\mathcal{L}\{x(t)\}$	ROC
$\delta(t)$	1	$s \in \mathbb{C}$
$\varepsilon(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\}>0$
$e^{-at}\varepsilon(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$-e^{-at}\varepsilon(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\}<\mathrm{Re}\{-a\}$
$t\varepsilon(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\}>0$
$t^n\varepsilon(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\}>0$
$te^{-at}\varepsilon(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$t^ne^{-at}\varepsilon(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\}>0$
$\cos(\omega_0 t)\varepsilon(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\}>0$
$e^{-at}\cos(\omega_0 t)\varepsilon(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$e^{-at}\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{-a\}$
$t\cos(\omega_0 t)\varepsilon(t)$	$\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\}>0$
$t\sin(\omega_0 t)\varepsilon(t)$	$\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\}>0$

Appendix B.2 Properties of the Bilateral Laplace Transform

	$x(t)$	$X(s)=\mathcal{L}\{x(t)\}$	ROC
Linearity	$Ax_1(t)+Bx_2(t)$	$AX_1(s)+BX_2(s)$	ROC $\supseteq$ ROC$\{X_1\}$ $\cap$ROC$\{X_2\}$
Delay	$x(t-\tau)$	$e^{-s\tau}X(s)$	not affected
Modulation	$e^{at}x(t)$	$X(s-a)$	Re$\{a\}$ shifted by Re$\{a\}$ to the right
'Multiplication by t', Differentiation in the frequency domain	$tx(t)$	$-\frac{d}{ds}X(s)$	not affected
Differentiation in the time domain	$\frac{d}{dt}x(t)$	$sX(s)$	ROC $\supseteq$ ROC$\{X\}$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	ROC $\supseteq$ ROC$\{X\}$ $\cap\{s:\mathrm{Re}\{s\}>0\}$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	ROC scaled by a factor of a

Appendix B.3 Fourier Transform Pairs

$x(t)$	$X(j\omega)=\mathcal{F}\{x(t)\}$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\dot{\delta}(t)$	$j\omega$
$\frac{1}{T}\perp\!\!\!\perp\!\!\!\perp\left(\frac{t}{T}\right)$	$\perp\!\!\!\perp\!\!\!\perp\left(\frac{\omega T}{2\pi}\right)$
$\varepsilon(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$\mathrm{rect}(at)$	$\frac{1}{\lvert a\rvert}\mathrm{si}\left(\frac{\omega}{2a}\right)$
$\mathrm{si}(at)$	$\frac{\pi}{\lvert a\rvert}\mathrm{rect}\left(\frac{\omega}{2a}\right)$
$\frac{1}{t}$	$-j\pi\,\mathrm{sign}(\omega)$
$\mathrm{sign}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$
$\sin(\omega_0 t)$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$
$e^{-\alpha\lvert t\rvert},\ \alpha>0$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$e^{-a^2t^2}$	$\frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}$

Appendix B.4 Properties of the Fourier Transform

	$x(t)$	$X(j\omega)=\mathcal{F}\{x(t)\}$
Linearity	$Ax_1(t)+Bx_2(t)$	$AX_1(j\omega)+BX_2(j\omega)$
Delay	$x(t-\tau)$	$e^{-j\omega\tau}X(j\omega)$
Modulation	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
'Multiplication by t' Differentiation in the frequency domain	$tx(t)$	$-\frac{dX(j\omega)}{d(j\omega)}$
Differentiation in the time domain	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$X(j\omega)\left[\pi\delta(\omega)+\frac{1}{j\omega}\right]$ $=\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right),\quad a\in\mathbb{R}\backslash\{0\}$
Convolution	$x_1(t)*x_2(t)$	$X_1(j\omega)\cdot X_2(j\omega)$
Multiplication	$x_1(t)\cdot x_2(t)$	$\frac{1}{2\pi}X_1(j\omega)*X_2(j\omega)$
Duality	$x_1(t)$ $x_2(jt)$	$x_2(j\omega)$ $2\pi x_1(-\omega)$
Symmetry relations	$x(-t)$ $x^*(t)$ $x^*(-t)$	$X(-j\omega)$ $X^*(-j\omega)$ $X^*(j\omega)$
Parseval theorem	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2\,dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2 d\omega$

Appendix B.6 Properties of the z-Transform

Property	$x[k]$	$X(z)$	ROC
Linearity	$ax_1[k]+bx_2[k]$	$aX_1(z)+bX_2(z)$	ROC $\supseteq$ ROC$\{X_1\}\cap$ROC$\{X_2\}$
Delay	$x[k-\kappa]$	$z^{-\kappa}X(z)$	ROC$\{x\}$; separate consideration of $z=0$ and $z\to\infty$
Modulation	$a^k x[k]$	$X\left(\frac{z}{a}\right)$	ROC$=\left\{z \,\middle\vert\, \frac{z}{a}\in\text{ROC}\{x\}\right\}$
Multiplication by k	$kx[k]$	$-z\frac{dX(z)}{dz}$	ROC$\{x\}$; separate consideration of $z=0$
Time inversion	$x[-k]$	$X(z^{-1})$	ROC$=\{z\,\vert z^{-1}\in\text{ROC}\{x\}\}$
Convolution	$x_1[k]*x_2[k]$	$X_1(z)\cdot X_2(z)$	ROC $\supseteq$ ROC$\{x_1\}\cap$ROC$\{x_2\}$
Multiplication	$x_1[k]\cdot x_2[k]$	$\frac{1}{2\pi j}\oint X_1(\zeta)X_2\left(\frac{z}{\zeta}\right)\frac{1}{\zeta}d\zeta$	multiply the limits of the ROC

Appendix B.5 Two-sided z-Transform Pairs

$x[k]$	$X(z)=\mathcal{Z}\{x[k]\}$	ROC
$\delta[k]$	1	$z\in\mathbb{C}$
$\varepsilon[k]$	$\frac{z}{z-1}$	$\vert z\vert>1$
$a^k\varepsilon[k]$	$\frac{z}{z-a}$	$\vert z\vert>\vert a\vert$
$-a^k\varepsilon[-k-1]$	$\frac{z}{z-a}$	$\vert z\vert<\vert a\vert$
$k\varepsilon[k]$	$\frac{z}{(z-1)^2}$	$\vert z\vert>1$
$ka^k\varepsilon[k]$	$\frac{az}{(z-a)^2}$	$\vert z\vert>\vert a\vert$
$\sin(\Omega_0 k)\varepsilon[k]$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$	$\vert z\vert>1$
$\cos(\Omega_0 k)\varepsilon[k]$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$	$\vert z\vert>1$