NTNU – Trondheim
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NTNU, Department of Physics

Solutions TFY4280 Signal Processing spring 2023

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Permitted examination aids according to code H:
No printed or hand-written support material is allowed
All calculators allowed

Problem 1

a) Kirchoff's laws yield

$$V_{\text{in}}(t) = \underline{\underline{L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(t')dt' + Ri(t)}}\ . \tag{1}$$

b) If we denote by $I(s)$ the Laplace transform of the current $i(t)$, the Laplace transform of the integro-differential equation with an input $V_{\text{in}}(t) = K\delta(t)$ is

$$LsI(s) + \frac{1}{Cs}I(s) + RI(s) = K\ . \tag{2}$$

Here K is a constant with dimension Volt $\times$ s. The unit impulse response function is $H(s) = RI(s)/K$ since $H(s)$ is dimensionless. We then find

$$H(s) = \frac{R}{L}\frac{s}{s^2 + \frac{R}{L}s + \frac{1}{CL}} = \underline{\underline{\frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}}}\ . \tag{3}$$

The engineering dimension of α and ω_0 is $(\text{time})^{-1}$.

c) Upon the substitution $s \to i\omega$, we find

$$H(i\omega) = -\frac{2i\alpha\omega}{\omega^2 - 2i\alpha\omega - \omega_0^2} \ . \tag{4}$$

Using $|z| = \sqrt{\bar{z}z}$ for a complex number z, we find

$$|H(i\omega)| = \underline{\underline{\frac{2\alpha\omega}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\alpha^2\omega^2}}}} \ . \tag{5}$$

From this equation, it is clear that $\omega = \omega_0$ maximizes $|H(i\omega)|$. [1] Moreover, $H(0) = 0$ and $\lim_{\omega\to\infty} H(\omega) = 0$. It is a band-pass filter with bandwidth α.

Problem 2

a) The output signal is

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\, d\tau \ . \tag{6}$$

In a causal system $y(t)$ only depends times $t' < t$. From the integral, above we see that this is possible only if the upper limit of the integral is t. Therefore $h(t-\tau) = 0$ for $\tau > t$ or $h(t) = 0$ for $t < 0$.

b) The integral above gives the inequality

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \right| \le \int_{-\infty}^{\infty} |x(\tau)||h(t-\tau)|d\tau \ . \tag{7}$$

If $|x(t)| < M \ \forall x$, then

$$|y(t)| \le M \int_{-\infty}^{\infty} |h(t-\tau)|d\tau \ . \tag{8}$$

The output is then bounded, i. e. the system is BIBO stable if the integral of $|h(t)|$ is finite ($h(t)$ is absolutely integrable).

c) The output signal is the convolution of $h(t)$ and the input signal,

$$y(t) = \underline{\underline{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}} \ . \tag{9}$$

[1] If you are not convinced, solve $\frac{d|H(i\omega)|}{d\omega} = 0$, the solution is $\omega = \omega_0$. Also show that the second derivative of $|H(i\omega)|$ is negative when evaluated at $\omega = \omega_0$.

d) The output signal $y(t)$ is now given by the integral

$$y(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)e^{-(t-\tau)}d\tau \ . \tag{10}$$

Using the properties of the step functions, we obtain

$$y(t) = u(t)\int_{0}^{t} e^{-(t-\tau)}d\tau = \underline{\underline{u(t)(1-e^{-t})}} \ . \tag{11}$$

Problem 3

a) The z-transform of the difference equation is

$$Y(z)\left[1 - z^{-1} + az^{-2}\right] = X(z) \ . \tag{12}$$

The discrete impulse response function is then

$$H(z) = \frac{Y(z)}{X(z)} = \underline{\underline{\frac{z^2}{z^2 - z + a}}} \ . \tag{13}$$

b) Stability is determined by the magnitude $|z|$ of the poles z of $H(z)$. The poles are

$$z = \frac{1 \pm \sqrt{1-4a}}{2} \ . \tag{14}$$

The system is stable if $|z| < 1$. The square root is imaginary for $a > \frac{1}{4}$. In this case

$$|z| = \sqrt{\frac{1}{4} + \frac{1}{4}(4a-1)} = a \ . \tag{15}$$

Stability then requires that $a < 1$. The square root is real for $a \leq \frac{1}{4}$. In this case, the positive root must satisfy

$$-1 < \frac{1 + \sqrt{1-4a}}{2} < 1 \ , \tag{16}$$

or $a > 0$. For the negative solution, we obtain

$$-1 < \frac{1 - \sqrt{1-4a}}{2} < 1 \ , \tag{17}$$

which is satisfied for $a > -2$. The system is therefore stable for

$$a \in \underline{\underline{(0,1)}} \ . \tag{18}$$

In Fig. 1, we show the magnitude of the two roots. The magnitude of the positive root is the blue line while the magnitude of the negative root is the light red line. Note that the magnitudes are equal for $a \geq \frac{1}{4}$, shown in dark red.

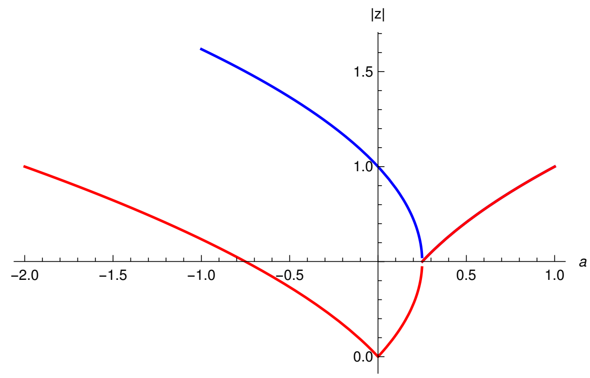

Figure 1: $|z|$ as a function of the parameter a.

c) For $a = 0$, the natural response satisfies the homogeneous difference equation

$$y_c[n] - y_c[n-1] \quad = \quad 0\ . \tag{19}$$

The solution is

$$y_c[n] \quad = \quad \underline{\underline{Cu[0]}}\ , \tag{20}$$

where C is an arbitrary constant. We note in passing that $y_c[n]$ does not vanish in the limit $n \to \infty$, which is in accordance with the fact that the system is not BIBO stable for $a = 0$. We next consider the forced response. The z-transform of the discrete unit step function is

$$X(z) \quad = \quad \frac{z}{z-1}\ . \tag{21}$$

This yields

$$Y(z) \quad = \quad H(z)X(z) = \frac{z^2}{(z-1)^2}\ . \tag{22}$$

The forced response $y_p[n]$ is then given by the inverse z-transform of Eq. (22). The inverse z-transform of the function $Y(z)$ is given by the convolution of the inverse of $H(z)$ and the inverse of $X(z)$. We notice that for $a = 0$, $X(z) = H(z)$ and $x[n] = h[n] = u[n]$. The particular solution is then given by the convolution

$$y_p[n] \quad = \quad x[n] * h[n] = \sum_{k=0}^{\infty} u[k]u[n-k] = \underline{\underline{n+1}}\ . \tag{23}$$

The complete solution is the sum of the natural response and the forced response $y[n] = y_c[n] + y_p[n]$. Taking into account the initial condition $y[0] = 1$, we find $C + 1 = 1$ or $C = 0$. This yields

$$y[n] \quad = \quad \underline{\underline{n+1}}\ . \tag{24}$$

Inserting $y[n] = n + 1$ into the difference equation with $a = 0$ shows that this indeed is a solution.

Problem 4

a) The function $f(x)$ is an even function. The coefficients B_k are proportional to the integral of $f(x)\sin(k\omega_0 x)$, which is an odd function on $[-1,1]$. They therefore vanish. Since the periodic extension of the function $f(x)$ is continuous, its Fourier series converges to $f(x)$ for all x.

b) The discrete-time signal $w[n]$ is the convolution of $x[n]$ and $y[n]$,

$$w[n] = \sum_{k=0}^{\infty} x[k]y[n-k] \,. \tag{25}$$

The z-transform of $w[n]$ is then

$$\begin{aligned} W(z) &= \sum_{n=0}^{\infty} w[n]z^{-n} = \sum_{k=0,n=k}^{\infty} x[k]y[n-k]z^{-n} \\ &= \sum_{k=0}^{\infty} x[k]z^{-k} \sum_{n=k}^{\infty} y[n-k]z^{-(n-k)} = \sum_{k=0}^{\infty} x[k]z^{-k} \sum_{n=0}^{\infty} y[n]z^{-n} \\ &= \underline{\underline{X(z)Y(z)}} \,, \end{aligned} \tag{26}$$

where we in the penultimate line have defined $n' = n - k$ and renamed n', $n' \to n$.