NTNU – Trondheim
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NTNU, Department of Physics

Solutions TFY4280 Signal Processing spring 2024

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Problem 1

a) For a general input signal $x(t)$, the output signal is

$$y(t) = \int_0^\infty u(t-\tau)x(\tau)\,d\tau = \underline{\underline{\int_0^t x(\tau)\,d\tau}}\ , \tag{1}$$

i.e. an integrator. The Laplace transform of the unit impulse response function is

$$H(s) = \int_{0^-}^\infty h(t)e^{-st}\,dt = \frac{1}{s}\ .$$

Since the pole of $H(s)$ is at $s=0$, the system is BIBO metastable.

b) The two unit impulse response functions $h(t)$ and $h_i(t)$ satisfy

$$h_i(t) * h(t) = \delta(t)\ . \tag{2}$$

This yields

$$\begin{aligned}\delta(t) &= \int_0^\infty h_i(\tau)u(t-\tau)\,d\tau \\ &= \int_0^t h_i(\tau)d\tau \ . \end{aligned} \tag{3}$$

Taking the derivative of this equation with respect to t yields the inverse impulse response function

$$h_i(t) = \underline{\underline{\delta'(t)}} \ . \tag{4}$$

Taking the derivative of Eq. (1) yields

$$x(t) = \underline{\underline{y'(t)}} \ ,$$

which shows that the inverse system is differentiator.

c) Using Kirchoff's laws we find

$$\underline{\underline{Ri(t) + \frac{1}{C}\int_0^t i(t')\,dt' = V_{\text{in}}(t)}} \ . \tag{5}$$

The engineering dimension of RC is time.

d) We denote by $I(s)$ and $V_{\text{in}}(s)$ the Laplace transform of the current $i(t)$ and the input voltage $V_{\text{in}}(t)$, respectively. This yields in the s-domain

$$RI(s) + \frac{1}{Cs}I(s) = V_{\text{in}}(s) \ , \tag{6}$$

or

$$I(s) = \frac{V_{\text{in}}(s)}{R + \frac{1}{Cs}} \ . \tag{7}$$

The Laplace transform of the voltage across the capacitor, $V_C(t)$, is

$$V_C(s) = \frac{1}{Cs}I(s) = \frac{V_{\text{in}}(s)}{1 + RCs} \ . \tag{8}$$

The transfer function $H(s)$ is the ratio between the output and the input, i.e. $H_C(s) = \frac{1}{1+RCs}$ and therefore

$$H_C(i\omega) = \underline{\underline{\frac{1}{1 + i\frac{\omega}{\omega_C}}}} \ , \tag{9}$$

where we have introduced the frequency $\omega_C = \frac{1}{RC}$. The magnitude becomes

$$|H_C(i\omega)| \;=\; \underline{\underline{\frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_C}\right)^2}}}}\;. \tag{10}$$

This is a first-order Butterworth filter, i.e. a low-pass filter.

e) This follows immediately from Eq. (8). The inverse Laplace transform of $V_C(s)$ is

$$V_C(t) \;\approx\; \underline{\underline{\frac{1}{RC}\int_0^{\infty} V_{\rm in}(t')\,dt'}}\;, \tag{11}$$

where we have used that the inverse Laplace transform of $\frac{F(s)}{s}$ is the integral of $f(t)$, where $f(t)$ is the inverse Laplace transform of $F(s)$. Thus the system is an integrator.

f) In this case we are interested in the ratio between $V_R(s)$, where $V_R(s)$ is the Laplace transform of the voltage across the resistor $VR(t)$. From the expression for $I(s)$, Eq. (7), we find

$$V_R(s) \;=\; RI(s) = \frac{V_{\rm in}(s)}{1+\frac{1}{RCs}}\;, \tag{12}$$

and therefore

$$H_R(s) \;=\; \underline{\underline{\frac{RCs}{1+RCs}}}\;. \tag{13}$$

This yields

$$|H_R(i\omega)| \;=\; \frac{\omega/\omega_c}{\sqrt{1+\left(\frac{\omega}{\omega_C}\right)^2}}\;. \tag{14}$$

This is a high-pass filter.

g) This follows directly from Eq. (12). Going back to the time-domain, the equation reads

$$V_R(t) \;\approx\; \underline{\underline{RCV_{\rm in}'(t)}}\;, \tag{15}$$

i.e. the system is a differentiator. Here we have used that the inversel Laplace transform of $sF(s)$ is $f'(t)$, where $f(t)$ is the inverse Laplace transform of $f(t)$.

h) The transfer functions are $H_C(s) = \frac{1}{1+RCs}$ and $H_R(s) = \frac{RCs}{1+RCs}$. The pole of these functions is located at $s = -\frac{1}{RC} < 0$, which implies the systems are BIBO stable. This is not in conflict with the result that an integrator is metastable since our system is only an integrator for $s \ll \frac{1}{RC}$. In this region $H_C(s) \approx 1$, but the approximation "ruins" the pole structure of $H_C(s)$.

Problem 2

The Fourier coefficients C_k are

$$\begin{aligned} C_k &= \frac{1}{T_0}\int_0^{T_0} e^{-ik\omega_0 t} t\, dt \\ &= \underline{\underline{\begin{cases} \frac{1}{2}T_0\ , & \text{for } k = 0, \\ \frac{iT_0}{2\pi k}\ , & \text{for } k \neq 0\ . \end{cases}}} \end{aligned} \tag{16}$$

The frequency spectrum is shown in Fig. 1 normalized by $T_0 = \frac{1}{2}$

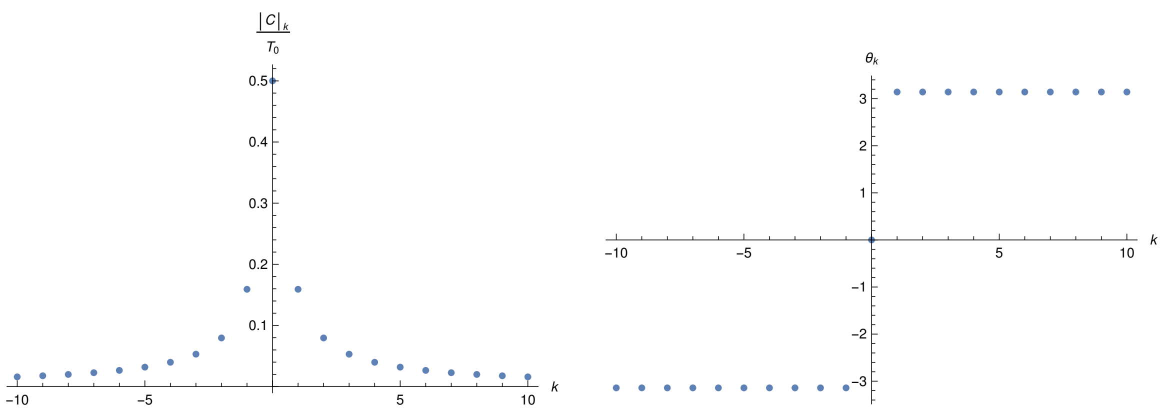

Figure 1: Frequency spectrum: Magnitude $|C_k|$ (left panel) and phase θ_k (right panel).

The Fourier series converges to the function $x(t)$ where $x(t)$ is continuous, i.e. for all values of x except $t = 0, \pm T_0 \ldots$. At these points, it converges to the average, i.e. to $\frac{1}{2}T_0$.

Problem 3

a) $H(z)$ is the z-transform of $h[n]$,

$$\begin{aligned} H(z) &= \sum_{k=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{k=0}^{\infty} (a/z)^n \\ &= \underline{\underline{\frac{1}{1 - a/z}}}\ , \end{aligned} \tag{17}$$

where we have used that the result for a geometric sum. The difference equation can be be read off from $H(z)$,

$$\underline{\underline{y[n] - ay[n-1] = x[n]}}\ . \tag{18}$$

b) The system is causal if $h[n] = 0$ for $n < 0$. This is the case **for all** a due to the unit step function.

c) The system is BIBO stable if

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty\ . \tag{19}$$

We therefore consider the sum

$$\sum_{k=-\infty}^{\infty} |h[k]| \quad = \quad \sum_{k=0}^{\infty} |a|^n = \frac{1}{1-|a|}\ . \tag{20}$$

This is finite for $|a| < 1$, i.e. for a inside the unit circle in the complex plane. Alternatively, consider the pole of $H(z)$, which is given by $1 - a/z = 0$ or $z = a$. The system is BIBO stable for $|z| < 1$ or $|a| < 1$.

d) From Eq. (18), it follows that the inverse system is given by

$$\underline{\underline{y[n] = x[n] - ax[n-1]}}\ . \tag{21}$$

In the z-domain, we know that $H(z)H_i(z) = 1$, where $H_i(z)$ is the z-transform of the inverse response function $h_i[n]$. This yields

$$H_i(z) \quad = \quad 1 - a/z\ . \tag{22}$$

The inverse z-transform of $H_i(z)$ yields

$$h_i[n] \quad = \quad \underline{\underline{\delta[n] - a\delta[n-1]}}\ . \tag{23}$$

Note that $h_i[n]$ is given by the response to the input $\delta[n]$. Using this input in Eq. (21) immediately gives Eq. (23).