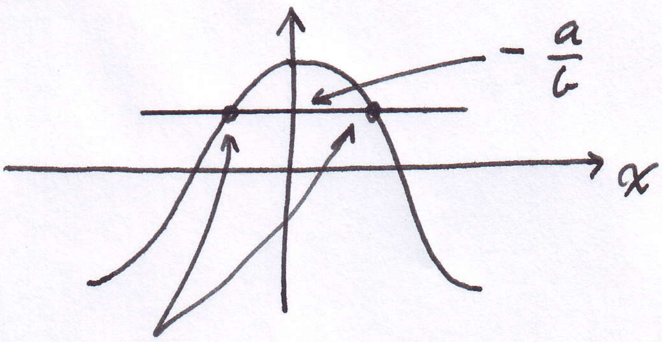
Solution to exam in TFY 4305, "Nonlinear Dynamics", December 12, 2008.

Problem 1

a) When $\underline{b > |a|}$, there are two fixed points given by $\cos x^* = -\frac{a}{b}$, one stable and one unstable:



Fixed points.

b) We set $f(x) = x + a + b\cos x$.

Bifurcations occur when

$$\underline{|f'(x^*)| = |1 - b\sin x^*| = 1.}$$

$\underline{\text{Saddle-node}}$ bifurcations occur when when $b = |a|$, where $x^* = \pi$ for $b = a$ (edge of region)

and $x^* = 0$ for $b = -a$ (edge of region)

– see figure on page 1.

c) The stable fixed point is characterized by $f'(x^*) = 1 - \sqrt{b^2 - a^2}$ whereas the unstable fixed point is characterized by $f'(x^*) = 1 + \sqrt{b^2 - a^2}$.

Only the <u>stable fixed point</u> may <u>bifurcate</u> in the interior of the region $-b < a < b$. This happens when

$$f'(x^*) = 1 - \sqrt{b^2 - a^2} = -1 \Rightarrow \underline{b = \sqrt{4 + a^2}}.$$

The bifurcation is a <u>period doubling</u>.

Problem 2

a) The fixed points are given by the equation set

$$\begin{cases} y^* = 0 \\ 2x^* = c(x^{*2} + a) \end{cases}$$

The second equation gives

$$\underline{x_\pm^* = \frac{1}{c}\left\{1 \pm \sqrt{1 - ac^2}\right\}}$$

which has real solutions if

$$\underline{ac^2 \le 1.}$$

For $\underline{ac^2 > 1}$, there are <u>no fixed points.</u>

b) The trace τ and determinant Δ of the Jacobian is

$$\underline{\tau = -b}\ , \quad \underline{\Delta = \frac{2(a - x^2)}{(x^2 + a)^2}}$$

Hence, τ is always negative and the sign of Δ is the sign of $a - x^2$.

By using that the fixed points $x_\pm^*$ are given by the equation $2x_\pm^* = (x_\pm^{*2} + a)c$, we find that

$$\underline{a - x_\pm^{*2}} = 2\left(a - \frac{1}{c} x_\pm^*\right) =$$

$$\underline{-\frac{2}{c^2}\left\{(1-ac^2) \pm \sqrt{1-ac^2}\right\}}$$

This implies that $\Delta < 0$ for x_+^* and $\Delta > 0$ for x_-^*, so that $\underline{x_+^* \text{ is a}}$ $\underline{\text{saddle point}}$ and $\underline{x_-^* \text{ is a node}}$ $\underline{\text{or a spiral (stable)}}$.

c) There is a $\underline{\text{saddle-node}}$ bifurcation at $\underline{ac^2 = 1}$.

d) $\ddot{x} = -\frac{2x}{x^2+a} + Cx = -\frac{d}{dx}\left\{\log(x^2+a) + Cx + \text{const}\right\}$

$\Rightarrow \underline{V(x) = \log(x^2+a) + Cx + \text{const.}}$

e) $\sqrt{a}$ is the <u>characteristic width of the potential</u>, b is the <u>damping</u> and c is a <u>constant external force.</u>

At the bifurcation point a stable equillibrium point is generated and the potential is able to capture a certain set of initial conditions.

f) Setting $b = c = 0$ and combining the two equations $\dot{x} = y$ and $\dot{y} = -\frac{2x}{x^2+a}$, we get

$$\ddot{x} + \frac{2}{a} x \frac{1}{1+\frac{x^2}{a}} = 0$$

Setting $z = \frac{x}{r}$ and $\tau = t\sqrt{2/a}$

we transform the equation into

$$z'' + \frac{z}{1+\epsilon z^2} = 0 \quad \text{where } \epsilon = \frac{r^2}{a}.$$

Assuming $\epsilon << 1$ and that $x \approx r$, we may expand this equation:

$$\underline{z'' + z - \epsilon z^3 = 0}$$

which is the <u>Duffing equation.</u>

We find the frequency of the periodic motion using the <u>two-timing</u> and <u>averaging technique.</u>

We assume two time scales $T = \epsilon\tau$ and $\zeta = \tau$, and expand the solution:

$$z(\tau) = z_0(T, \zeta) + \epsilon z_1(T, \zeta) + \cdots$$

The two-timing substitutions into 7

the equation gives

$$\partial_{\zeta\zeta} z_0 + z_0 = 0 \qquad O(1)$$

$$\partial_{\zeta\zeta} z_1 + z_1 = -2\partial_{\zeta T} z_0 + z_0^3 \quad O(\epsilon)$$

We now assume that

$$z_0 = \tilde{r}(T)\cos(\zeta + \phi(T))$$

$$= \tilde{r}(T)\cos\theta(\zeta, T)$$

This gives

$$-2\partial_{\zeta T} z_0 + z_0^3 = 2(\tilde{r}'\sin\theta + \tilde{r}\phi'\cos\theta)$$
$$+ \tilde{r}^3\cos^3\theta$$

The secular terms have to be

zero to avoid blow-ups:

$$\left.\begin{aligned} 2\tilde{r}' - b_1 &= 0 \\ 2\tilde{r}\phi' - a_1 &= 0 \end{aligned}\right\}$$

where $a_1 = -2\tilde{r}^3 \langle \cos^3\theta \cos\theta \rangle = -\frac{3}{4}\tilde{r}^3$

and $b_1 = -2\tilde{r}^3 \langle \cos^3\theta \sin\theta \rangle = 0$ from

Fourier analysis.

This implies $\underline{\tilde{r}' = 0}$ and $\underline{\phi' = -\frac{3}{8}\tilde{r}^2}$.

Since $z = x/r$, we set $\tilde{r} = 1$, so

that $\underline{\phi' = -\frac{3}{8}}$.

The angular frequency is given by

$$\underline{\omega} = \frac{d\theta}{dt} = \frac{d\tau}{dt}\,\frac{d\theta}{d\tau} = \sqrt{\frac{2}{a}}\Big(1 + \phi' \underbrace{\frac{d\tau}{d\tau}}_{\epsilon}\Big)$$

$$= \sqrt{\frac{2}{a}}\left(1 - \frac{3}{8}\epsilon\right)$$

$$= \underline{\underline{\sqrt{\frac{2}{a}}\left(1 - \frac{3}{8}\frac{r^2}{a}\right)}}$$