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Final exam in TFY4305

Dynamical systems

December 10, 2010.

Solution set

Problem 1:

a) $(x^*, y^*) = \begin{cases} (2,3) \\ (-1,0) \\ (+1,0) \end{cases}$

b) $J = \begin{pmatrix} 2x & -1 \\ y & (x-2) \end{pmatrix}$

c) $(2,3)$: $J = \begin{pmatrix} 4 & -1 \\ 3 & 0 \end{pmatrix}$

$\tau = 4$, $\Delta = 3$ $\Rightarrow$ unstable node

②

$(-1,0)$ $\quad J = \begin{pmatrix} -2 & -1 \\ 0 & -3 \end{pmatrix}$

$\tau = -5,\ \Delta = 6 \Rightarrow$ stable node

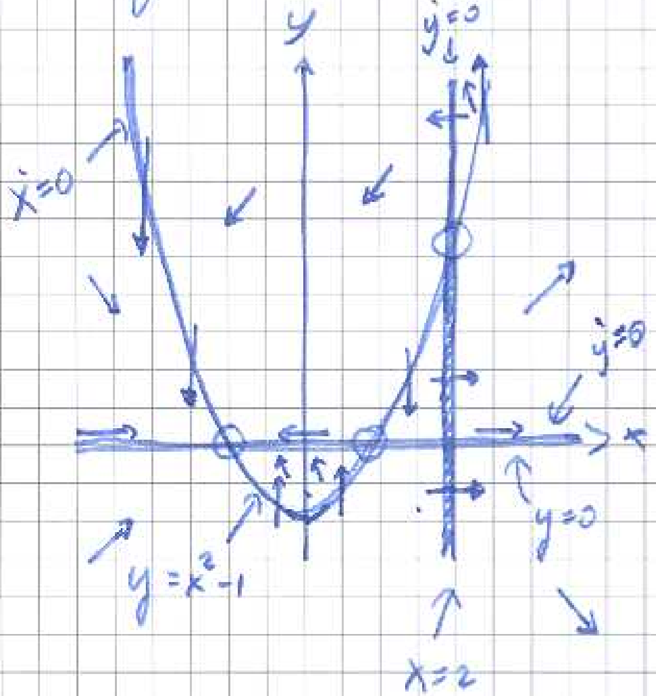
$(1,0)$ $\quad J = \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix}$

$\tau = 1,\ \Delta = -2 \Rightarrow$ saddle point.

d)

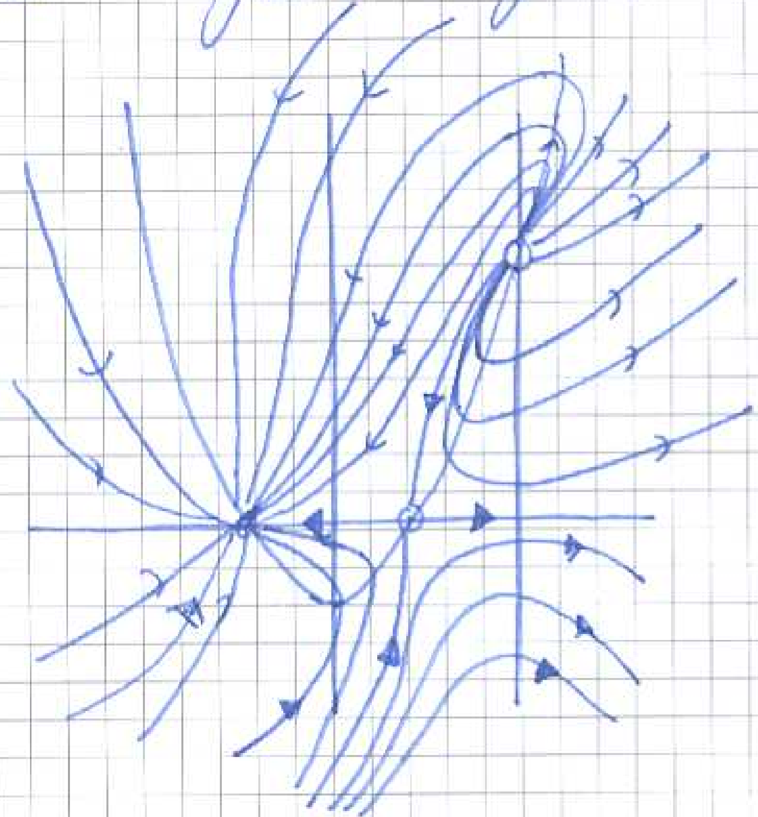
$\dot{x} = 0$ nullcline: $\quad y = x^2 - 1$

$\dot{y} = 0$ nullclines: $\quad \left.\begin{matrix} y = 0 \\ x = 2 \end{matrix}\right\}$



③ e)

The phase portrait.



Problem 2

a) $x^* = 0, \; \underline{1 \pm \sqrt{1-r^2}}$

Only exists for $-1 \leq r \leq 1$

b) Bifurcations take place at

$r_{c1} = 1$ and $r_{c2} = -1$

c) Both bifurcations are saddle-node bifurcations

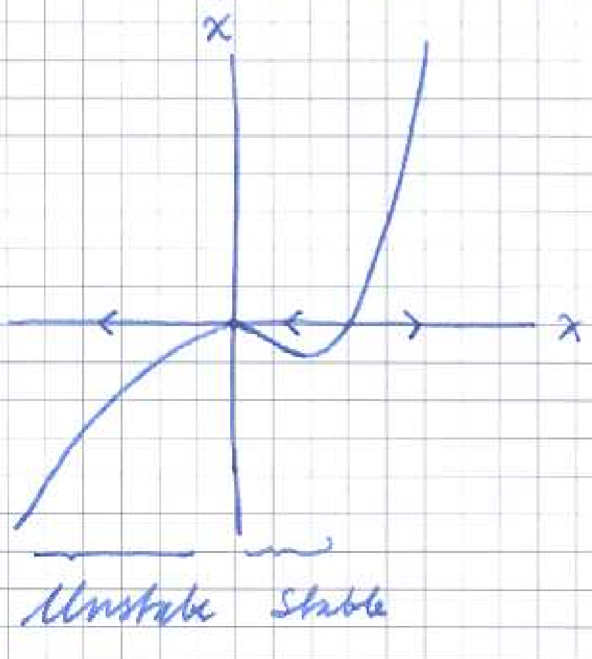
④ d)

$$f'(x) = r^2 - 4x + 3x^2$$

$\underline{x^* = 0:}$ $\quad f'(0) = r^2$

$r \neq 0$: Unstable

$r = 0$: Draw the curve:



$\underline{x^* = 1 - \sqrt{1 - r^2}}$

For $r = 0$, this saddle point coincides with $x^* = 0$.

$f'(1 - \sqrt{1-r^2}) \leq 0$ for $-1 < r < 1$

with equality for $r = 0$.

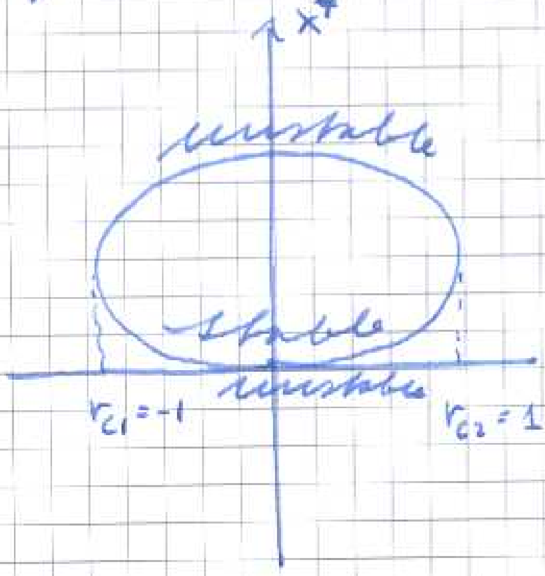
$\Rightarrow$ Stable fixed point.

⑤ $x^* = 1 + \sqrt{1-r^2}$

$f'(x^* = 1+\sqrt{1-r^2}) > 0$ for $-1 < r < 1$

$\Longrightarrow$ Unstable node

e) Bifurcation diagram



Problem 3

a) $a = \frac{1}{r}$ $b = \frac{1}{2}$

$$R = \frac{1}{4}r^2 - \frac{1}{2}r$$

The transformation is thus:

$$x_n = \frac{1}{r}\tilde{x}_n + \frac{1}{2}$$

⑥

b)

$$x_0 = R - (R - x_0^2)^2$$

$$= R - R^2 - x_0^4 + 2Rx_0^2$$

c)

$$f_R^{2\,\prime}(x_0) = f_R'(x_0) \cdot f_R'(f(x_0))$$

$$= (-2x_0)(-2(R - x_0^2))$$

$$= 4x_0 R - 4x_0^3$$

d) A period-doubling bifurcation occurs when $f_R^{2\,\prime}(x_0) = -1 \Rightarrow$

$$\underline{x_0^3 - x_0 R - \tfrac{1}{4} = 0}$$