**Department of physics, NTNU**
**TFY4340/FY8909 Nanophysics**

# Exam, 24 May 2017

1. **Various qualitative questions.**

   Use only a few sentences to answer each question.

   a. What is the difference between Bloch states and surface states?

   b. Weak localization in a diffusive conductor: Why does the conductivity increase when a weak magnetic field is applied to the sample? How could one use weak localization to estimate the phase coherence length $l_\varphi$?

   c. Assuming isotropic conditions (i.e., there is no directional dependence of any relevant property), how is the electron effective mass $m^*$ defined in terms of the electronic dispersion $E(k)$?

2. **Drude formula.**

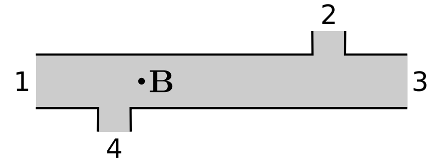
   Derive the Drude formula relating the current density to the applied electric field,

$$\mathbf{j} = \frac{e^2 n \tau}{m^*}\mathbf{E}, \tag{1}$$

   where $n$ is the electron density and $\tau$ is the average time an electron has traveled freely since its last scattering from an impurity.

3. **Landauer-Büttiker formalism and the quantum Hall effect.**

   Consider the following ideal 4-terminal device:




   A strong uniform magnetic field **B** is applied perpendicular to the 2DEG and points out of the plane.

The Landauer-Büttiker equations,

$$I_\alpha = \sum_{\beta \neq \alpha} G_{\alpha\beta}(V_\alpha - V_\beta), \tag{2}$$

with conductances

$$G_{\alpha\beta} = \frac{2e^2}{h} T_{\alpha\beta}, \tag{3}$$

relate the net current entering into terminal $\alpha$ to the potentials at the various terminals. Here, $T_{\alpha\beta}$ denotes the "direct transmission sum" from terminal $\beta$ to terminal $\alpha$.

a. Assume that only the lowest Landau level lies below the Fermi energy $E_\mathrm{F}$ in the bulk region of the 2DEG. Express the relation between currents and potentials as

$$I_\alpha = \frac{2e^2}{h} \sum_{\beta=1}^{4} \gamma_{\alpha\beta} V_\beta, \tag{4}$$

and write down the $4 \times 4$ matrix $\gamma$.

b. Let terminals 1 and 3 be the "source" and the "drain" respectively, whereas terminals 2 and 4 are ideal voltage probes. Find the Hall resistance $R_\mathrm{H} = R_{13,24} = (V_2 - V_4)/I_1$ and the 2-terminal resistance $R_{2t} = R_{13,13} = (V_1 - V_3)/I_1$.

c. Interchange the roles of terminals 2 and 3 and find the 2-terminal resistance $R_{2t} = R_{12,12} = (V_1 - V_2)/I_1$ and the longitudinal resistance $R_L = R_{12,34} = (V_3 - V_4)/I_1$.

d. Qualitatively: With terminals 1 and 3 as source and drain, and terminals 2 and 4 as voltage probes (as in b), explain how $R_\mathrm{H}$ will change as we decrease the magnetic field strength. Explain also how $R_L$ depends on $B$ when terminals 3 and 4 are used as voltage probes (as in c).

4. **Fano factor of a diffusive conductor.**

For a diffusive conductor (e.g. a wire with many impurities, causing many scattering events, resulting in diffusive motion of the electrons in the wire) it turns out that the distribution function of the transmission values $P(x)$ does not depend on the details of the design, and reads

$$P(x) = \left\langle \sum_n \delta(x - T_n) \right\rangle = \frac{\langle G \rangle}{2G_c} \frac{1}{x\sqrt{1-x}}, \tag{5}$$

where $\{T_n\}$ is the set of transmission probabilities through the conductor and $G_c = 2e^2/h$ is the contact conductance.

a. Using this distribution function, show that $\langle G \rangle$ equals the expectation value of the conductance at low temperatures $k_B T \ll eV$.

b. Calculate the expectation value of the shot-noise power, which is the zero-temperature-limit of the noise power,

$$\langle S_{\rm sn} \rangle = \langle S(0) \rangle \big|_{k_B T \to 0} = \left\langle 2 \frac{\langle\langle Q^2 \rangle\rangle}{\Delta t} \right\rangle \Bigg|_{k_B T \to 0}, \tag{6}$$

where $Q$ is the transmitted charge in the time interval $\Delta t$.

c. What is the expected Fano factor

$$\langle F \rangle = \frac{\langle S_{\rm sn} \rangle}{2e \langle I \rangle}, \tag{7}$$

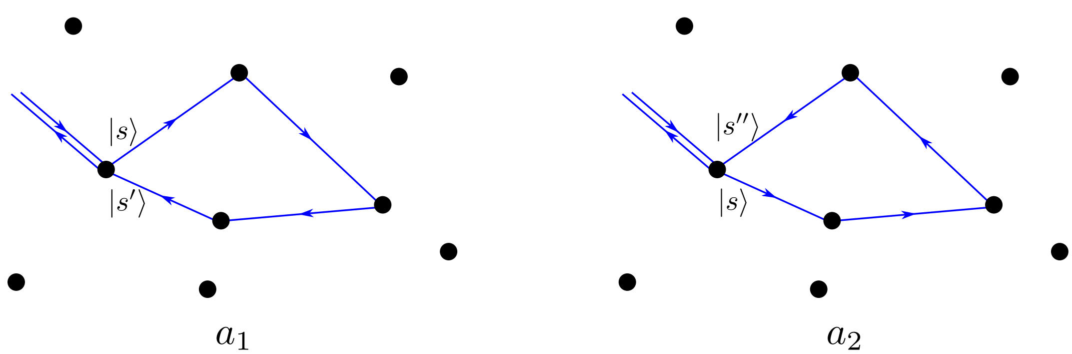
for a diffusive conductor?

Hint: If you forgot the explicit expression for $S_{\rm sn}$, you can derive it from the cumulant-generating function

$$\ln \Lambda(\chi) = 2\Delta t \int \frac{dE}{h} \sum_n \ln \Big\{ 1 + T_n(E)[e^{i\chi} - 1] f_{\rm S}(E)[1 - f_{\rm D}(E)] + T_n(E)[e^{-i\chi} - 1] f_{\rm D}(E)[1 - f_{\rm S}(E)] \Big\}. \tag{8}$$

5. **Weak antilocalization.**

We consider electronic propagation in a diffusive metal with strong spin-orbit interaction. To understand localization effects due to phase coherent propagation along closed loops, we consider two time-reversed paths $a_1$ and $a_2$ along a closed loop that contribute to exact backscattering, such as schematically sketched below.




We assume that all scattering is elastic, that the propagation along both paths is fully phase coherent, and that there is no magnetic field applied.

The spin-orbit interaction couples the momentum of an electron to its spin, and therefore the spin state of the electrons is changed while they travel through the material. For an electron traveling along a given closed loop (such as $a_1$), its initial spin state

$$|s\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{with } |a|^2 + |b|^2 = 1, \tag{9}$$

is related to its final spin state $|s'\rangle$ through some unitary transformation,

$$|s'\rangle = \hat{U}_r \, |s\rangle \,, \tag{10}$$

which we can interpret as a rotation in spin space and takes the form of a $2 \times 2$ matrix in our spinor notation. Most generally, we can decompose this rotation as

$$\hat{U}_r = \hat{R}_z(\alpha)\hat{R}_y(\theta)\hat{R}_z(\beta), \tag{11}$$

in terms of three angles $\alpha$, $\beta$, and $\theta$. The operator $\hat{R}_k(\phi)$ denotes a rotation of $\phi$ along the $k$-axis in spin space, and from the lecture on electron spin we remember that

$$\hat{R}_k(\phi) = e^{-\frac{i}{\hbar}\phi \hat{S}_k}, \tag{12}$$

with

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and } \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

a. Show that

$$\hat{R}_z(\phi) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}, \tag{14}$$

$$\hat{R}_y(\phi) = \begin{pmatrix} \cos\phi/2 & -\sin\phi/2 \\ \sin\phi/2 & \cos\phi/2 \end{pmatrix}. \tag{15}$$

Hints: (i) The explicit result of a matrix exponential is given by its power expansion. (ii) The following two power expansions could be helpful:

$$\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}, \tag{16}$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}. \tag{17}$$

b. Give the explicit $2 \times 2$ matrix form of $\hat{U}_r$ as a function of $\alpha$, $\beta$, and $\theta$.

c. Since the time-dependent electron's momentum is exactly opposite along the time-reversed path $a_2$ (compared to $a_1$), the spin rotation acquired along $a_2$ is also exactly opposite. Show that for the overlap of the two resulting spin states (after traveling along $a_1$ and $a_2$) one finds

$$\langle s''|s'\rangle = \langle s|\hat{U}_r^2|s\rangle. \tag{18}$$

d. There are many localization loops in a mesoscopic sample, all resulting in a different $\hat{U}_r$. We assume that the spin-orbit interaction is so strong that for each loop of interest the actual $\alpha$, $\beta$, and $\theta$ can be considered as being drawn randomly from a uniform distribution. Calculate the expectation value for $\langle s''|s'\rangle$.

   Hint: $\sin x \cos x = \frac{1}{2} \sin 2x$.

d. Explain qualitatively how this result reflects in the conductivity of the sample, as compared to a similar sample without spin-orbit interaction.