Department of physics, NTNU
TFY4340/FY8909 Nanophysics

Exam, 7 June 2018

1. **Various qualitative questions.**

   Use only a few sentences to answer each question.

   a. Explain what distinguishes a metal, an insulator, and a semiconductor, in terms of the electronic band structure.

   b. Explain what a Schottky barrier is.

   c. Describe the Aharonov-Bohm effect and explain qualitatively how it arises.

2. **Chemical potential for a two-dimensional electron gas.**

   a. Calculate the density of states

   $$D_2(E) = \frac{dN}{dE},$$

   with $N(E)$ the number of allowed states with energy E or smaller, for a two-dimensional free electron gas contained in an area A.

   b. Express the chemical potential μ at temperature T in terms of the total electron density n.
   Hint:

   $$\int_0^\infty dx\, \frac{1}{ae^x + 1} = \ln\left(1 + \tfrac{1}{a}\right)$$

3. **Statistics of rare electron transfers**

   We consider the statistics of electron transfers through a tunnel junction where all transmission probabilities $T_n \ll 1$ are small. This means that we can assume all successful electron transfers through the junction to be uncorrelated, which allows us to find a simple expression for the characteristic function describing the transfers.

   We start by considering a very short time interval dt, during which the chance for a transfer is $dt/\tau \ll 1$, where $1/\tau$ is the transfer rate.

a. Show that the characteristic function of the probability distribution $P_{N,dt}$ for counting N transfers within time dt can be approximated

$$\Lambda_{dt}(\chi) = \langle e^{i\chi N} \rangle \approx \exp\left\{dt(e^{i\chi} - 1)/\tau\right\}.$$

b. Assuming all transfers to be uncorrelated, write down the characteristic function $\Lambda_{\Delta t}(\chi)$ for a larger time interval Δt in terms of the average number of transfers $\langle N \rangle = \Delta t/\tau$.

c. Show that the distribution function $P_{N,\Delta t}$ describing the probability to have N transfers occurring in the time interval Δt is a Poisson distribution,

$$P_{N,\Delta t} = e^{-\langle N \rangle} \frac{\langle N \rangle^N}{N!}.$$

d. Find the probability to have no transfers in a time interval Δt.

e. Calculate the second cumulant $\langle\langle N^2 \rangle\rangle_{\Delta t}$ for the time interval Δt.

f. Find the Fano factor

$$F = \frac{\langle\langle N^2 \rangle\rangle_{\Delta t}}{\langle\langle N \rangle\rangle_{\Delta t}},$$

for a tunnel junction in this Poisson limit.

4. **Joule heating in the Drude model.**

The Drude model uses the assumption that all collisions between electrons and impurities randomize the direction of motion of the electron, $\langle v_{x,y,z} \rangle = 0$ directly after a collision. Another assumption, which we did not discuss in the lectures, is that the collisions are *inelastic*: It is assumed that each collision reduces the kinetic energy of the electron to the same value $\frac{1}{2}mv_i^2$ (which is related to the temperature of the sample). Indeed, if we would have assumed elastic collisions, the kinetic energy of the electrons would constantly increase in the presence of an electric field, growing almost indefinitely for large samples.

a. We assume an electric field E_x present, which points in the x-direction. As we know, the effect of this field is to shift v_x by an amount $-eE_xt/m$ during a time t. Calculate the gain in kinetic energy during time t

$$\Delta E(t) = \frac{1}{2}m\left[v(t)^2 - v(0)^2\right],$$

if the initial velocity is $\mathbf{v}(0) = v_i(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Show that assuming the *direction* of $\mathbf{v}(0)$ to be random yields an average energy gain during time t

$$\langle \Delta E(t) \rangle = \frac{e^2 E_x^2 t^2}{2m}.$$

b. The Drude model assumes all collisions with impurities to be uncorrelated. This means that the statistics of these collisions are exactly the same as those of the electron transfers through a tunnel junction considered in problem 3, and we can thus use the results we obtained there: If the probability for an electron to collide with an impurity in an infinitesimal time interval dt is dt/τ, where $1/\tau$ is the collision rate, then the probability that a randomly picked electron has suffered no collisions during the preceding time t is $e^{-t/\tau}$.

   Argue why the probability density function for the time between two successive collisions for a single electron reads

$$p(t) = \frac{1}{\tau} e^{-t/\tau}.$$

   Hint: In terms of actual probabilities, $p(t)dt$ is the probability to find a time between two collisions in the interval $[t, t + dt]$.

c. Calculate the average time between two collisions.

d. Use the results from (a) and (b) to calculate the average energy an electron gains between two collisions.

e. Assuming that all this energy gained between two collisions is transfered to the lattice at the second collision, show that the average energy transfer from the electrons to the sample per volume per second reads

$$\sigma E_x^2,$$

   where $\sigma = ne^2\tau/m$ is the conductivity, with n being the electron density.

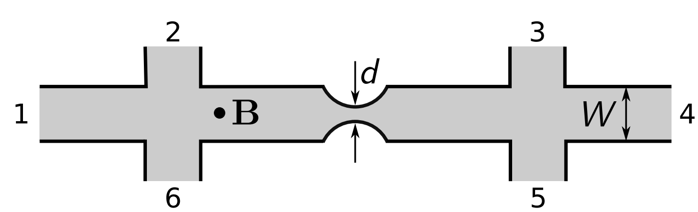
f. Show that this yields for the total energy dissipated per second in a conductive sample of length L and cross section A the familiar result

$$P = I^2 R,$$

   where $R = L/\sigma A$ is the total resistance of the sample.

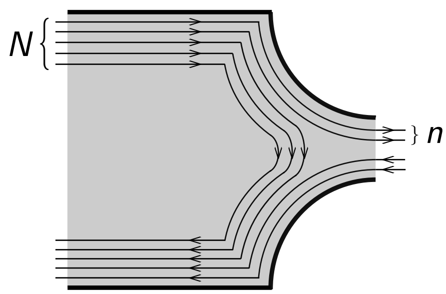
5. **Landauer-Büttiker formalism and the quantum Hall effect.**

   Consider the following 6-terminal Hall bar:



The bar has a width W, but in the middle there is a constriction of width d. A uniform magnetic field $\mathbf{B}$ is applied perpendicularly to the 2DEG and points out of the plane, leading to N edge states in the wide regions of the device. Inside the constriction there are only $n < N$ edge states available at the Fermi level. We thus assume that there are n channels with perfect transmission in each direction through the constriction; the other $N - n$ edge states in the left and right parts of the device are not connected.

The following zoom-in might help to visualize the configuration of the edge channels near the constriction, where I picked $N = 5$ and $n = 2$:



The Landauer-Büttiker equations,

$$I_\alpha = \sum_{\beta \neq \alpha} G_{\alpha\beta}(V_\alpha - V_\beta),$$

with conductances $G_{\alpha\beta} = (2e^2/h)T_{\alpha\beta}$ relate the net current entering into terminal α to the potentials at the various terminals. Here, $T_{\alpha\beta}$ denotes the "direct transmission sum" from terminal β to terminal α.

a. Neglecting impurity scattering and tunneling processes, list all non-zero $T_{\alpha\beta}$ and give their magnitudes.

b. Let terminals 1 and 4 be the "source" and the "drain" respectively through which a current I runs, whereas the other terminals are ideal voltage probes. Find
   - the Hall resistance $R_{14,35} = (V_3 - V_5)/I$;
   - the Hall resistance $R_{14,26} = (V_2 - V_6)/I$;
   - the Hall resistance $R_{14,25} = (V_2 - V_5)/I$;
   - the Hall resistance $R_{14,36} = (V_3 - V_6)/I$;
   - the longitudinal resistance $R_{14,23} = (V_2 - V_3)/I$;
   - the longitudinal resistance $R_{14,65} = (V_6 - V_5)/I$;
   - the 2-terminal resistance $R_{14,14} = (V_1 - V_4)/I$.

   For convenience you can set $V_4 = 0$.

   Advice: This looks like a lot of contacts to consider and resistances to calculate. But don't panic, just write down the correct equations for all the I_α and you will see that everything is rather simple to solve.