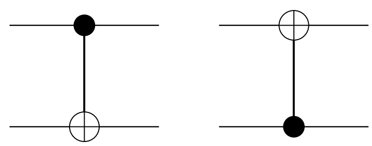
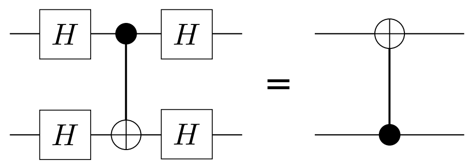
The CNOT gate.

(a) Consider the CNOT and "inverted" CNOT gates as depicted below:



Give the representation of both gates in terms of 4×4 matrices, written in the same basis $\mathcal{H}_1 \otimes \mathcal{H}_2 = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ where $\mathcal{H}_1$ is the Hilbert space spanned by the first (upper) qubit and $\mathcal{H}_2$ that of the second (lower) qubit.

(b) Prove the equality



where H represents the single-qubit Hadamard gate,

$$H = i\hat{R}_{\mathbf{h}}(\pi),$$

with $\hat{R}_{\mathbf{h}}(\theta) = \exp(-\frac{i}{2}\theta\, \mathbf{h}\cdot\hat{\boldsymbol{\sigma}})$ the rotation operator implementing a rotation over the angle θ around the Hadamard axis $\mathbf{h} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{z})$.

Scattering matrix and transfer matrix.

(a) We consider a general scattering matrix in the usual form

$$\hat{s} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix},$$

where all four elements are matrices themselves as well. Unitarity of $\hat{s}$ ensures that both $\hat{s}\hat{s}^\dagger = 1$ and $\hat{s}^\dagger\hat{s} = 1$. Use these conditions to show that you can write

$$t = (t^\dagger)^{-1} + r'(t')^{-1}r.$$

(b) Use the definition of the scattering matrix,

$$\begin{pmatrix} \mathbf{b}_L \\ \mathbf{b}_R \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} \mathbf{a}_L \\ \mathbf{a}_R \end{pmatrix},$$

to derive the transfer matrix $\hat{m}$,

$$\begin{pmatrix} \mathbf{b}_R \\ \mathbf{a}_R \end{pmatrix} = \hat{m} \begin{pmatrix} \mathbf{a}_L \\ \mathbf{b}_L \end{pmatrix},$$

that connects the amplitudes on the two sides of the scatterer rather than outgoing to incoming amplitudes.

Give the four elements of $\hat{m}$ explicitly, in terms of t, r, t', and r'.

Transmission and Landauer-Büttiker conductance.

(a) We assume that a particle with mass m and energy ϵ moves in one strictly one-dimensional channel in the presence of a single scatterer with a potential

$$V(x) = V_0\,\delta(x),$$

where $\delta(x)$ is the Dirac-delta function and V_0 characterizes the strength of the scatterer.

As we do in Sec. 4.3 of the Lecture Notes, we write the wave function in the regions $x < 0$ (denoted L) and $x > 0$ (denoted R) as

$$\psi_L(x) = \frac{1}{\sqrt{2\pi\hbar v_L}}\left[a_L e^{ikx} + b_L e^{-ikx}\right], \qquad \text{for } x < 0,$$
$$\psi_R(x) = \frac{1}{\sqrt{2\pi\hbar v_R}}\left[a_R e^{-ikx} + b_R e^{ikx}\right], \qquad \text{for } x > 0,$$

where there is no sum over channels since we assume a strictly one-dimensional system.

We define a scattering matrix consisting of reflection r (r') and transmission amplitudes t (t') for electrons coming from the left (right) toward the scatterer as

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}.$$

Consider the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0\,\delta(x)\right]\psi(x) = \epsilon\,\psi(x),$$

and show that

$$r = r' = \frac{V_0}{i\hbar v - V_0},$$
$$t = t' = \frac{i\hbar v}{i\hbar v - V_0},$$

where $v = \sqrt{2\epsilon/m} = \hbar k/m$ is the same on both sides, since the potential vanishes there.

(b) Imagine that we have two such scatterers in series in the one-dimensional channel, located at $x = -a/2$ and $x = a/2$ with a total potential

$$V(x) = V_0\,\delta(x + a/2) + V_0\,\delta(x - a/2)\,.$$

As a scattering problem, we can now view the system as consisting of three scattering regions in series where the leftmost and rightmost scattering matrices are identical to the one in (a), $S_1 = S_3 = S$, and in the middle region of the system ($-a/2 < x < a/2$), there is no reflection, but the transmission amplitude acquires a dynamical phase $e^{\pm ika}$ so that its scattering matrix is

$$S_2 = \begin{pmatrix} 0 & e^{ika} \\ e^{ika} & 0 \end{pmatrix}.$$

Calculate the total transmission *amplitude* for the whole system (consisting of scattering matrices S_1, S_2, and S_3).

Hint: Use the result from the previous problem *Scattering matrix and transfer matrix.* This simplifies the calculation, but the answer can also be found without using that result.

(c) Show that the total transmission *probability* can be written as

$$T_{\text{tot}} = \frac{T^2}{1 - 2R\cos\xi + R^2}$$

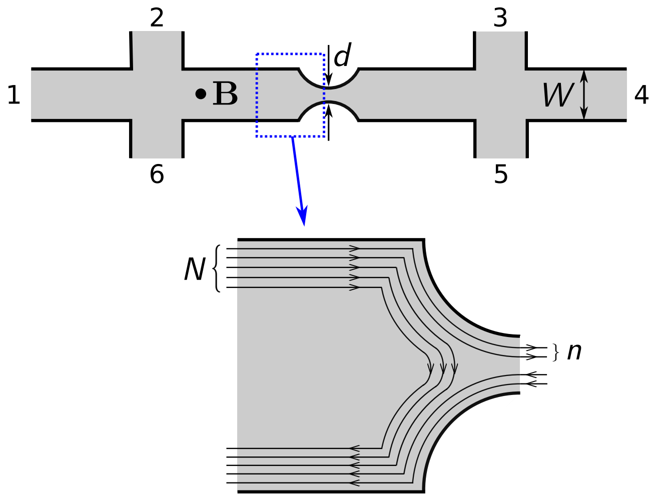
where $\xi = 2ka + 2\arctan \hbar v/V_0$.

(d) Assume that the transmission probability T for each scatterer is very small $T \ll 1$. What is the maximum transmission probability $T_{\text{tot}}(\epsilon)$ and the associated Landauer-Büttiker conductance G?

Comment on the physical interpretation of this result.

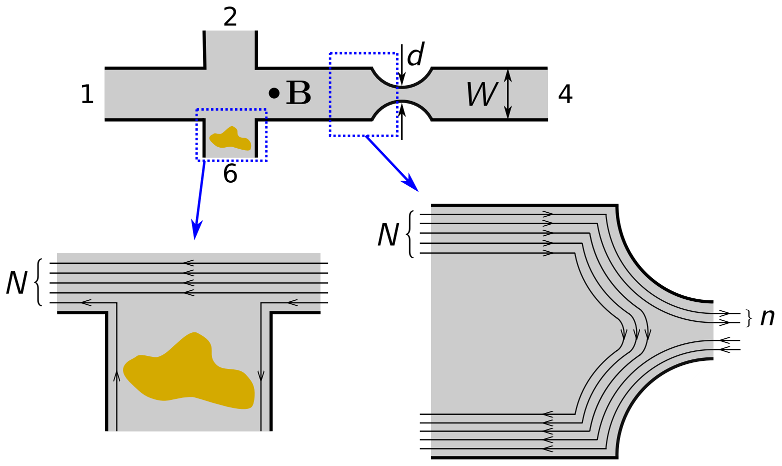
Quantum Hall Effect.

(a) In the "test exam" we discussed during the lectures, we considered a six-terminal Hall bar as depicted below:



We assumed that terminals 2, 3, 5, and 6 were pure voltage probes, and thus that $I_2 = I_3 = I_5 = I_6 = 0$. In the center of the Hall bar there was a constriction that was so narrow that only n out of the N edge states could propagate from left to right, and vice versa. On of the results we found was that the Hall resistances $R_{14,26}$ and $R_{14,35}$ both equaled $h/2Ne^2$, i.e., they were not affected by the backscattering caused by the constriction (they don't depend on n).

We modify this setup as follows: We disregard terminals 3 and 5, and we assume that there is a defect or impurity *inside* terminal 6, so that only the *outermost* edge state (the state associated with the $n = 0$ Landau level) reaches the voltage probe, as shown below:



Assuming that $N > n$ and $n \geq 1$, write down the Landauer-Büttiker conductance matrix G, that relates the four currents to the four voltages,

$$\begin{pmatrix} i_1 \\ i_2 \\ i_4 \\ i_6 \end{pmatrix} = G \begin{pmatrix} V_1 \\ V_2 \\ V_4 \\ V_6 \end{pmatrix},$$

using the renormalized currents $i_\alpha = hI_\alpha/2e^2$.

(b) Assuming again that terminals 2 and 6 are perfect voltage probes and setting $I_1 = I$, calculate the Hall resistance

$$R_{\mathrm{H}} = \frac{V_2 - V_6}{I}.$$