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SOLUTION TO EXAM
TFY4340 MESOSCOPIC PHYSICS
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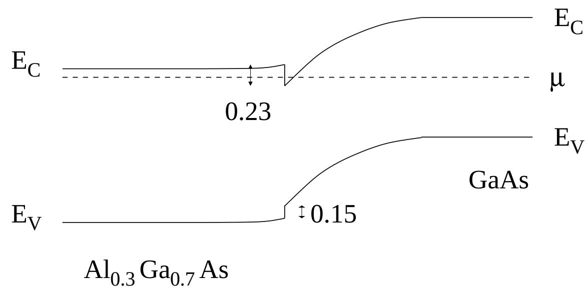
QUESTION 1

a) Imagine contact between n–AlGaAs and GaAs is established with each material in separate equilibria. Concentration gradients will result in diffusion of electrons from n–AlGaAs into GaAs and diffusion of holes in the other direction. The resulting electric dipole causes a built–in electric field $\boldsymbol{E}$ and potential V, and therefore band bending near the interface. The equilibrium condition is a common Fermi level μ. Far from the interface, the situation is unchanged.



b) Introduction of an undoped layer of AlGaAs will increase the distance between the 2DEG and the donor impurities in n–AlGaAs. This will reduce the elastic scattering rate and increase the mean free path for the electrons in the 2DEG. The mobility of the 2DEG will increase.

c) The Schrödinger equation for $\Phi_n(z)$ is, with potential $V(z) = Fz$,

$$-\frac{\hbar^2}{2m^*}\frac{d^2\Phi_n}{dz^2} + Fz\Phi_n = E\Phi_n.$$

Division by $-F$ and multiplication with κ yields

$$\frac{1}{\kappa^2}\frac{d^2\Phi_n}{dz^2} - \kappa z\Phi_n = -\frac{E\kappa}{F}\Phi_n$$

when using the given definition of κ. Introduction of $\xi = \kappa z$ and $\tilde{E} = E\kappa/F$ then yields the given equation

$$\frac{d^2\Phi_n}{d\xi^2} - \xi\Phi_n = -\tilde{E}\Phi_n.$$

d) Subband n starts at energy

$$E_n = \frac{F\tilde{E}_n}{\kappa}.$$

The constant $1/\kappa$ is a length with the value

$$\left(\frac{(1.05\cdot 10^{-34})^2}{2\cdot 0.067\cdot 9.1\cdot 10^{-31}\cdot 10\cdot 10^{-3}\cdot 1.6\cdot 10^{-19}/10^{-9}}\right)^{1/3} = 3.837\,\text{nm}.$$

Hence, subband 1 starts at energy

$$E_1 = 10\cdot 2.34\cdot 3.837 = 90\,\text{meV},$$

and subband 2 starts at energy

$$E_2 = 10\cdot 4.09\cdot 3.837 = 157\,\text{meV}.$$

With a Fermi level at $\mu = 100$ meV, it is clear that only the lowest 2D subband will be occupied by electrons.

e) Solution of the Schrödinger equation in 2D, with periodic boundary conditions (PBC), yields plane–wave solutions

$$\psi(x, y) \sim e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

with allowed values of the wave vector,

$$\boldsymbol{k} = k_x\hat{x} + k_y\hat{y} = \frac{2\pi}{L}\left(n_1\hat{x} + n_2\hat{y}\right).$$

Hence, there is 1 allowed value of $\boldsymbol{k}$ in an area $(2\pi/L)^2$ in k–space, and, because of spin degeneracy $g_S = 2$, there are 2 allowed states within this k–space area. Hence, the DOS in k–space is constant,

$$D_2(\boldsymbol{k}) = \frac{2}{(2\pi/L)^2} = \frac{L^2}{2\pi^2}.$$

f) The region of k–space with absolute value of the wave vector smaller than a given k is in 2D a disk with area πk^2. With the constant DOS derived above, the number of states within this disk is

$$N_2(k) = \pi k^2 \cdot \frac{L^2}{2\pi^2} = \frac{L^2 k^2}{2\pi}.$$

g) Since $E(k) = \hbar^2 k^2/2m^*$, i.e., $k = \sqrt{2m^* E/\hbar^2}$, the number of states with energy less than E is

$$N_2(E) = \frac{L^2}{2\pi} \cdot \frac{2m^* E}{\hbar^2} = \frac{m^* L^2}{\pi\hbar^2} E.$$

Therefore, the 2D DOS is

$$D_2(E) = \frac{dN_2}{dE} = \frac{m^* L^2}{\pi\hbar^2},$$

a constant, independent of the energy.

h) $-e$ has unit C, dE has unit J, $\rho_j^+(E)$ has unit 1/Jm, $v_j(E)$ has unit m/s, and $T_j(E)$ is dimensionless. Hence, the product of all of these factors has the unit C/s, i.e., A.

i) Contribution from right–going states in subband j:

$$I_j^+ = (-e)\int_{E_j^t}^{\mu_1} dE\, \rho_j^+(E)\, v_j(E)\, T_j(E).$$

Contribution from left–going states in subband j:

$$I_j^- = -(-e)\int_{E_j^t}^{\mu_2} dE\, \rho_j^-(E)\, v_j(E)\, T_j(E).$$

The total 1D DOS pr unit length in subband j is

$$\rho_j(E) = \frac{D_1(E)}{L} = \frac{\sqrt{2m^*}}{\pi\hbar\sqrt{E}}.$$

This is distributed equally between right– and left–going states, so

$$\rho_j^+(E) = \rho_j^-(E) = \frac{1}{2}\rho_j(E).$$

Therefore, the total current due to subband j is

$$I_j = I_j^+ + I_j^- = (-e)\int_{\mu_2}^{\mu_1} dE\, \rho_j^+(E)\, v_j(E)\, T_j(E).$$

The product of the 1D DOS pr unit length and the group velocity is simply a constant,

$$\rho_j^+(E)\, v_j(E) = \frac{\sqrt{m^*}}{\pi\hbar\sqrt{2E}} \cdot \sqrt{2E/m^*} = \frac{1}{\pi\hbar}.$$

Hence, in the linear response limit $\mu_1 \simeq \mu_2 \simeq E_F$, we obtain

$$I_j = (-e)(\mu_1 - \mu_2)T_j(E_F)/\pi\hbar = (-e)(-eV)T_j(E_F)/\pi\hbar = \frac{2e^2}{h} T_j(E_F)\, V,$$

with $V = (\mu_1 - \mu_2)/(-e)$ the applied voltage between S and D. If several 1D subbands have their bottom below the Fermi level E_F, we must sum up the current contribution from each of them. Consequently,

$$G = \frac{I}{V} = \frac{\sum_j I_j}{V} = \frac{2e^2}{h}\sum_j T_j(E_F).$$

j) The DOS pr unit area of the 2DEG is

$$\frac{D_2(E)}{L^2} = \frac{m^*}{\pi\hbar^2},$$

where we used the result in 1**g)**. With a constant DOS pr unit area, we have simply

$$n_2 = \frac{D_2}{L^2} \cdot E_F,$$

so the Fermi level in the 2DEG is

$$E_F = \frac{n_2\pi\hbar^2}{m^*} = \frac{3.56 \cdot 10^{15} \cdot \pi \cdot 1.05^2 \cdot 10^{-68}}{0.067 \cdot 9.1 \cdot 10^{-31}} \simeq 13\,\text{meV}.$$

k) From the figure, we read off a conductance of $6 \cdot 2e^2/h$ at a gate voltage -1.5 V. This means that 6 1D subbands have energy below the Fermi level E_F. (However, subband nr 7 has higher energy than E_F.) The energy levels in a 1D potential box of width W are

$$E_j = \frac{p_j^2}{2m^*} = \frac{\hbar^2 k_j^2}{2m^*} = \frac{\hbar^2(2\pi/\lambda_j)^2}{2m^*} = \frac{2\hbar^2\pi^2}{m^*\lambda_j^2}.$$

The boundary condition on the wave functions, $\psi_j = 0$ at both ends of the 1D box, gives us the possible wavelengths as

$$\lambda_1 = 2W,\ \ \lambda_2 = W,\ \ \ldots\ \ \lambda_j = \frac{2W}{j},$$

and the energy levels

$$E_j = \frac{\hbar^2 \pi^2 j^2}{2m^* W^2} \quad (j = 1, 2, 3, \ldots).$$

If we take $E_j = E_F$, we find the channel width

$$W = \frac{\hbar \pi j}{\sqrt{2m^* E_F}} = \frac{1.05 \cdot 10^{-34} \pi j}{\sqrt{2 \cdot 0.067 \cdot 9.1 \cdot 10^{-31} \cdot 13 \cdot 10^{-3} \cdot 1.6 \cdot 10^{-19}}} \simeq 21j \text{ nm}.$$

In our case, E_F lies somewhere between E_6 and E_7, so the channel width W lies somewhere between 126 and 147 nm, e.g., at about 135 nm.

QUESTION 2

a) With the given information, we have the following transmission sums:

$$\begin{aligned} T_{21} = T_{15} = T_{43} &= N \\ T_{52} = T_{34} &= N - n \\ T_{32} = T_{54} &= n \end{aligned}$$

and all the others are zero. (Actually, we should perhaps have interchanged all these indices, since negatively charged electrons will be deflected to the right, with the given direction of the magnetic field.)

b) We write down the Büttiker–Landauer equations (with $i_\alpha = hI_\alpha/2e^2$ and $i = hI/2e^2$, to save some typing):

$$\begin{aligned} i = i_1 &= T_{15}(V_1 - V_5) = N(V_1 - V_5) \\ 0 = i_2 &= T_{21}(V_2 - V_1) = N(V_2 - V_1) \\ 0 = i_3 &= T_{32}(V_3 - V_2) + T_{34}(V_3 - V_4) = n(V_3 - V_2) + (N - n)(V_3 - V_4) = -nV_2 + NV_3 \\ -i = i_4 &= T_{43}(V_4 - V_3) = N(V_4 - V_3) = -NV_3 \\ 0 = i_5 &= T_{52}(V_5 - V_2) + T_{54}(V_5 - V_4) = (N - n)(V_5 - V_2) + n(V_5 - V_4) = -(N - n)V_2 + NV_5 \end{aligned}$$

From the 2. equation, we have $V_2 = V_1$, which, when inserted into the 3. equation yields $V_3 = nV_1/N$, and when inserted into the 5. equation yields $V_5 = (N - n)V_1/N$. The 4. equation gives $I = (2e^2/h)NV_3 = (2e^2/h)nV_1$. The various resistances are now

$$\begin{aligned} R_{14,23} &= \frac{V_2 - V_3}{I} = \frac{h}{2e^2}\frac{1 - n/N}{n} = \frac{h}{2e^2}(1/n - 1/N) \\ R_{14,25} &= \frac{V_2 - V_5}{I} = \frac{h}{2e^2}\frac{1 - (N - n)/N}{n} = \frac{h}{2e^2}(1/N) \\ R_{14,35} &= \frac{V_3 - V_5}{I} = \frac{h}{2e^2}\frac{n/N - (N - n)/N}{n} = \frac{h}{2e^2}(2/N - 1/n) \\ R_{14,14} &= \frac{V_1 - V_4}{I} = \frac{h}{2e^2}\frac{1 - 0}{n} = \frac{h}{2e^2}(1/n) \end{aligned}$$

QUESTION 3

a) The system is in the diffusive but phase–coherent regime, i.e., $L \gg l_e$ and $L \ll l_\phi$. Here, L is the system size, l_e is the average distance between elastic scatterers, and l_ϕ is the phase coherence length. In zero magnetic field, there will now be an increased probability of precise back–scattering, since a given (diffusive) path that results in precise back–scattering and its time–reversed counterpart will interfere constructively. High probability of back–scattering means high resistance.

b) If we turn on a weak magnetic field, the constructive interference described in **a)** will be destroyed, the resistance will go down, and the conductance (or conductivity) will increase.

c) The argument in **a)** requires phase–coherent transport. Increasing the temperature will reduce the phase–coherence length, and eventually wipe out the weak localization effect.