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SOLUTION TO EXAM
TFY4340 MESOSCOPIC PHYSICS
Wednesday May 18 2011, 0900 - 1300

QUESTION 1

a) The 6 corners of the 1BZ - the so-called K points - are located at $\pm(4\pi/3\sqrt{3}a)\hat{y}$, $\pm(b_{2x}\hat{x} + (1/3)b_{2y}\hat{y}) = \pm((2\pi/3a)\hat{x} + (2\pi/3\sqrt{3}a)\hat{y}$, and $\pm(b_{1x}\hat{x} + (1/3)b_{1y}\hat{y}) = \pm((2\pi/3a)\hat{x} - (2\pi/3\sqrt{3}a)\hat{y}$. Here, it is easiest to calculate $E^{\pm}$ at the corner point on the k_y axis, where we have

$$\cos\frac{\sqrt{3}k_y a}{2} = \cos\frac{2\pi}{3} = -\frac{1}{2}.$$

Hence,

$$E^{\pm} = E_0 \pm |\gamma|\sqrt{1 + 4\cdot 1\cdot\left(-\frac{1}{2}\right) + 4\cdot\left(-\frac{1}{2}\right)^2} = E_0.$$

This shows that the valence band E^- and the conduction band E^+ have equal energies at the six corners of the 1BZ. In other words, the bandgap is zero.

b) We take the given hint and look at e.g. $E^{\pm}(k_x, 4\pi/3\sqrt{3}a)$ for small values of k_x. We then have

$$\cos\frac{3k_x a}{2} \simeq 1 - \frac{1}{2}\left(\frac{3k_x a}{2}\right)^2 = 1 - \frac{9k_x^2a^2}{8},$$

and

$$\begin{aligned} E^{\pm} &\simeq E_0 \pm |\gamma|\sqrt{1 + 4\cdot\left(1 - \frac{9k_x^2a^2}{8}\right)\cdot\left(-\frac{1}{2}\right) + 1} \\ &= E_0 \pm |\gamma|\sqrt{\frac{9k_x^2a^2}{4}} \\ &= E_0 \pm \frac{3|\gamma||k_x|a}{2} \end{aligned}$$

Photons are massless particles, so the energy–momentum relation for photons is simply $E = pc$. For electrons in a crystal, we know that the wavenumber, measured relative to the energy band minimum, and multiplied by $\hbar$, is the crystal momentum. More precisely, in our situation,

$$p = \hbar k_x,$$

so we may write for the graphene electrons at the Fermi level, near the conduction band minimum,

$$E = \frac{3}{2}|\gamma|ak_x = v_F p,$$

with Fermi velocity

$$v_F = \frac{3|\gamma|a}{2\hbar}.$$

The nearest neighbour distance in graphene is $a \simeq 1.4$ Å, so if $\gamma \sim 2$ eV, the velocity of the electrons is

$$v_F \sim \frac{3\cdot 2\cdot 1.6\cdot 10^{-19}\cdot 1.4\cdot 10^{-10}}{2\cdot 1.05\cdot 10^{-34}} \simeq 640 \text{ km/s}.$$

QUESTION 2

With zero applied magnetic field, the spin up and the spin down electrons both see one magnetic layer with parallel and one with antiparallel magnetization. Hence, the resistance for both types of spin is $R_+ + R_-$. In the two-current model, the total resistance $R(H = 0)$ is obtained by treating the two spins as separate and independent conducting channels, i.e., their individual resistances are added as resistors in parallel:

$$R(H = 0) = \left(\frac{1}{R_+ + R_-} + \frac{1}{R_+ + R_-} \right)^{-1} = \frac{R_+ + R_-}{2}.$$

With an applied magnetic field $H > H_s$, one half of the electrons sees two magnetic layers with parallel magnetization, whereas the other half sees two layers with antiparallel magnetization. Hence, the resistance for one half of the electrons is $2R_+$ and for the other half, it is $2R_-$. In this case, the total resistance is therefore

$$R(H > H_s) = \left(\frac{1}{2R_+} + \frac{1}{2R_-} \right)^{-1} = \frac{2R_+R_-}{R_+ + R_-}.$$

The ratio of the latter to the former is about 0.55, or 11/20, in the experiment by Baibich et al. Hence, the opposite ratio is

$$\frac{R(H = 0)}{R(H > H_s)} = \frac{(R_+ + R_-)^2}{4R_+R_-} = \frac{20}{11}.$$

In terms of the ratio $x = R_+/R_-$, this yields the equation

$$x/4 + 1/4x + 1/2 - 20/11 = 0,$$

or

$$x^2 - 58x/11 + 1 = 0,$$

which has the solution ($x > 1$ is specified in the exam question)

$$x = 29/11 + \sqrt{(58/11)^2 - 4}/2 \simeq 5.1.$$

QUESTION 3

a) The bandgap in GaAs is smaller than in $Al_{0.3}Ga_{0.7}As$, by about 0.4 eV. The lattice constant is almost the same in the two materials (they are "lattice matched"), so thin layers (down to a few atomic layers) of one of the materials can be grown on top of the other without causing much strain at the interface. The different bandgaps result in a lower conduction band edge (and a higher valence band edge) in GaAs than in $Al_{0.3}Ga_{0.7}As$. The DBS potential energy profile represents the position dependent conduction band minimum, where the two potential barriers are layers of $Al_{0.3}Ga_{0.7}As$ and the rest is GaAs.

b) To leading order when $E \ll V_0$ and $\kappa d \gg 1$, a number of simplifications may be done: $\sigma \simeq \kappa/k \gg 1$, $\delta \simeq \kappa/k \simeq \sigma \gg 1$, and $\sinh \kappa d \simeq \cosh \kappa d \simeq \exp(\kappa d)/2$. In the expression for t, we may then neglect $\cosh \kappa d$ in comparison with $i(\delta/2)\sinh \kappa d$ (since $\delta \gg 1$), and we obtain

$$T = |t|^2 \simeq \frac{4}{\delta^2 \sinh^2 \kappa d} \simeq \frac{16k^2}{\kappa^2}\, e^{-2\kappa d}.$$

Since $\sigma \simeq \delta$, it follows immediately that $R = |r|^2 \simeq 1$ in the opaque barrier limit, as expected.

c) The amplitude of the path in the left figure is $t \cdot \exp(ikL) \cdot t$, i.e., transmission through the left barrier, free propagation across the well, and finally transmission through the right barrier. The amplitude of the path in the right figure contains an additional factor $r \cdot \exp(ikL) \cdot r \cdot \exp(ikL)$, due to reflection inside the well off the right barrier, free propagation across the well (to the left), reflection off the left barrier, and free propagation across the well (to the right). The next path involves four reflections inside the well, and hence yet another factor $r^2 \exp(2ikL)$. And so on! This is a geometric series:

$$\begin{aligned} t_{SD} &= t^2 e^{ikL} + t^2 e^{ikL} r^2 e^{2ikL} + t^2 e^{ikL} r^4 e^{4ikL} + \ldots \\ &= t^2 e^{ikL} \sum_{j=0}^{\infty} \left(r^2 e^{2ikL} \right)^j \\ &= \frac{t^2 e^{ikL}}{1 - r^2 e^{2ikL}}, \end{aligned}$$

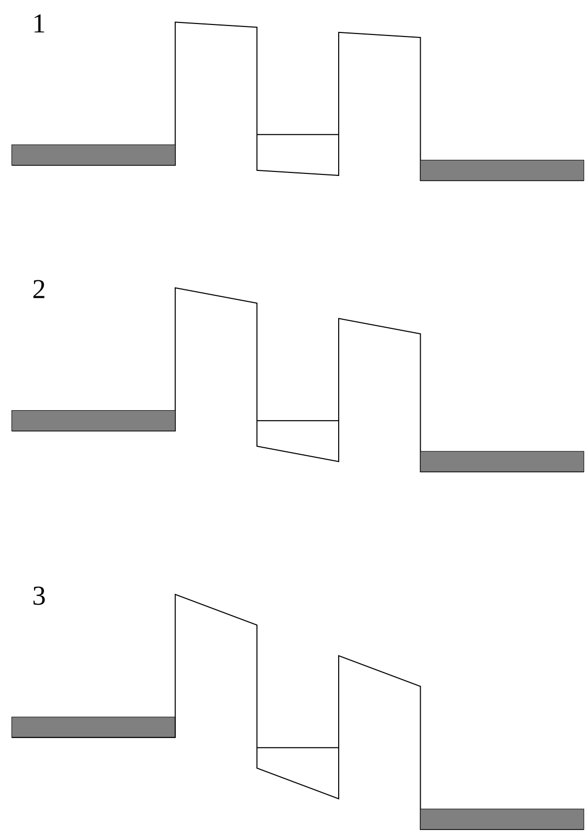
which is what we were asked to show.

d) If we neglect the phase of the reflection amplitude, the transmission probability of the DBS may be written (using $R = 1 - T$ where necessary)

$$\begin{aligned} T_{SD} &= |t_{SD}|^2 \simeq \frac{T^2}{|1 - Re^{2ikL}|^2} \\ &= \frac{T^2}{(1 - R\cos 2kL)^2 + R^2 \sin^2 2kL} \\ &= \frac{T^2}{1 - 2R\cos 2kL + R^2} \\ &= \frac{T^2}{1 - 2(1-T)\cos 2kL + (1-T)^2} \\ &= \frac{T^2}{1 - 2\cos 2kL + 2T\cos 2kL + 1 - 2T + T^2} \\ &= \frac{T^2}{2(1-T)(1 - \cos 2kL) + T^2} \\ &\simeq \frac{T^2}{2(1 - \cos 2kL) + T^2} \end{aligned}$$

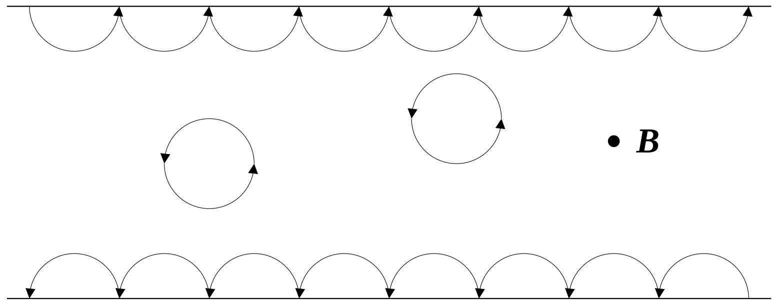
The first term in the denominator disappears when $\cos 2kL = 1$, giving resonant tunneling $T_{SD} = 1$ for $2kL = n \cdot 2\pi$ ($n = 1, 2, \ldots$), i.e., for electron waves with $\lambda = 2\pi/k = 2L/n$. This is precisely the condition for standing waves inside the well of width L.

e) The current is essentially given by an integral of the transmission probability, over energies corresponding to states that are occupied in the source contact and empty in the drain contact. When the applied voltage V is increased from zero, the resonances are typically all located above the Fermi level, and the resulting current is small (figure 1 below). At a certain value of V, a resonance falls below the Fermi level of the source contact, and the current increases significantly due to the large transmission probability at resonance (figure 2 below). A further increase in V brings this resonance below the conduction band edge of the source contact, and the current drops (figure 3 below). In this region, dI/dV is negative, and we have an $I(V)$ curve with a region of negative differential resistance.



QUESTION 4

a) Classically, the electrons in the 2DEG will move in circular orbits with radius $r = r_c = v/\omega_c = mv/eB$ (the cyclotron radius). With $\boldsymbol{B}$ pointing out of the plane, the electrons will be deflected to the left. Far from the edges of the 2DEG, the electrons are able to complete the circular orbit, and they will not contribute to the net current I along the channel (between terminals 1 and 4). Electrons located near the upper or lower edge of the 2DEG will not be able to complete the circular orbit. They will collide with the edge and move in "skipping orbits" along the edge - to the left near the lower edge and to the right near the upper edge:



b) An electron entering the 2DEG in terminal 3 will move along the upper edge and end up in terminal 4. The edge states near the lower edge are far away, so the probability of an electron being scattered from a right-going edge state near the upper edge to a left-going edge state near the lower edge is very small. (Deviations from this idealized situation are discussed in **d** below.) Hence, only elements $T_{j+1,j}$ (and T_{15}) will be non-zero, and with N edge states at the Fermi level near each of the two edges we have

$$T_{21} = T_{32} = T_{43} = T_{54} = T_{15} = N.$$

All the other elements of the transmission matrix are zero.

c) The terminals 2, 3, and 5 are voltage probes, so $I_2 = I_3 = I_5 = 0$. From the Büttiker–Landauer equations we then have

$$\begin{aligned} I_1 &= G_{15}(V_1 - V_5) = I, \\ I_2 &= G_{21}(V_2 - V_1) = 0, \\ I_3 &= G_{32}(V_3 - V_2) = 0, \\ I_4 &= G_{43}(V_4 - V_3) = -I, \\ I_5 &= G_{54}(V_5 - V_4) = 0. \end{aligned}$$

The 2. and 3. of these equations immediately yield $V_3 = V_2 = V_1 = V$. The 5. equation yields $V_5 = V_4 = 0$. Hence,

$$\begin{aligned} R_H &= R_{14,25} = \frac{V_2 - V_5}{I} = \frac{V}{I} = \frac{1}{G_{15}} = \frac{h}{2e^2}\frac{1}{N} \\ R_L &= R_{14,23} = \frac{V_2 - V_3}{I} = 0 \end{aligned}$$

d) The results obtained in **c)** are valid as long as the Fermi level E_F lies well above the energy $(N-1/2)\hbar\omega_c$ of the N-th Landau level in the bulk region of the 2DEG, and well below the energy $(N+1/2)\hbar\omega_c$ of the $N-1$-th Landau level in the bulk region. In that case, back-scattering is negligible, and the Hall resistance is quantized, whereas the longitudinal resistance is zero. This is clearly observed in the experiment. For example, when $8 < B < 9.5$ T, there is a Hall plateau with

$$\rho_{xy}(B) = \frac{E_y}{j_x} = \frac{V_H}{I} = R_H(B)$$

at a value slightly below 13 kΩ. This corresponds well with the value of the "resistance quantum" $h/2e^2 \simeq 12.9$ kΩ. In this magnetic field range, we see that

$$\rho_{xx}(B) = \frac{E_x}{j_x} = \frac{V_2 - V_3}{I}\frac{W}{L} = R_L(B)\frac{W}{L}$$

is zero, in agreement with **c)**. For such a strong magnetic field, only one bulk-region Landau level lies below the Fermi level, giving rise to electron transport via a single edge state. Reduction of B to about 4 T brings a second Landau level below E_F, transport takes place via two edge states, and the Hall resistance is reduced to about 6.4 kΩ. Again, we see that the longitudinal resistance is essentially zero.

In the transition between two Hall plateaus, the highest occupied Landau level will at some stage have an energy very close to the Fermi level. This state will then be extended across the whole sample in the transverse direction, and the probability of back-scattering increases severely. This explains why the longitudinal resistance R_L, or the longitudinal resistivity ρ_{xx}, increases significantly. The oscillations in R_L as a function of B are called Shubnikov - de Haas oscillations. Disorder (e.g. impurities) introduces localized states within the 2DEG, and scattering from one edge to the other becomes more effective than in a perfect sample.