NORGES TEKNISK-
NATURVITENSKAPELIGE UNIVERSITET
INSTITUTT FOR FYSIKK

Contact during the exam:
Jon Andreas Støvneng
Phone: 73 59 36 63 / 45 45 55 33

SOLUTION TO EXAM
TFY4340 MESOSCOPIC PHYSICS
Friday June 1 2012, 0900 - 1300

QUESTION 1

Einstein – Photoelectric effect – 1921
Fert/Grünberg – Giant magnetoresistance – 2007
Geim/Novoselov – Graphene – 2010
Giæver/Esaki/Josephson – Tunneling phenomena – 1973
von Klitzing – Quantum Hall effect – 1985

QUESTION 2

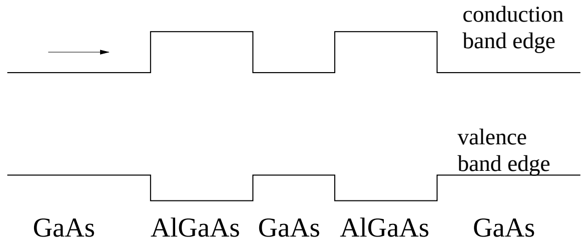
a) Bloch states are *extended* states, given as a product of a plane wave and a function periodic in the lattice. Surface states are *localized* to the surface(s) of the system.

b) Mesoscopic physics is physics on a length scale intermediate between atomic length scales and macroscopic length scales, typically of the order 10 – 100 nm. The system size L is comparable to or smaller than various characteristic length scales, such as the elastic mean free path, phase coherence length, Fermi wavelength etc.

c) $m^* = \hbar^2 (d^2E(k)/dk^2)^{-1}$. (Use $E = p^2/2m^* = \hbar^2 k^2/2m^*$ to remember the details.)

d) A metal has one or more partially filled bands. An insulator has completely filled and completely empty bands only.

e) The two semiconductors have different bandgaps, so by growing e.g. GaAs – AlGaAs – GaAs – AlGaAs – GaAs, we end up with a double barrier structure for electrons in the conduction band coming in e.g. from the left:



For certain energies of the incoming electron, corresponding typically to standing waves in the GaAs well region, the transmission probability is high, and we have resonant tunneling.

QUESTION 3

a) The total transmitted electron wave is a superposition of the two waves ψ_L and ψ_R corresponding to traversing the gold ring in the left and right semicircle, respectively:

$$\psi_T = \psi_L + \psi_R.$$

With this symmetric setup, the only difference between ψ_L and ψ_R is the quantum phase acquired due to the vector potential:

$$\begin{aligned}
\psi_T &= \psi_0 \left[\exp\left(-\frac{ie}{\hbar} \int_L \boldsymbol{A} \cdot d\boldsymbol{l} \right) + \exp\left(-\frac{ie}{\hbar} \int_R \boldsymbol{A} \cdot d\boldsymbol{l} \right) \right] \\
&= \psi_0 \exp\left(-\frac{ie}{\hbar} \int_L \boldsymbol{A} \cdot d\boldsymbol{l} \right) \left[1 + \exp\left(\frac{ie}{\hbar} \left(\int_L - \int_R \right) \boldsymbol{A} \cdot d\boldsymbol{l} \right) \right] \\
&= \psi_0 \exp(-i\phi_L) \left[1 + \exp\left(\frac{ie}{\hbar} \oint \boldsymbol{A} \cdot d\boldsymbol{l} \right) \right] \\
&= \psi_0 \exp(-i\phi_L) \left[1 + \exp\left(\frac{ie}{\hbar} \int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{S} \right) \right] \\
&= \psi_0 \exp(-i\phi_L) \left[1 + \exp\left(\frac{ie}{\hbar} \int \boldsymbol{B} \cdot d\boldsymbol{S} \right) \right] \\
&= \psi_0 \exp(-i\phi_L) \left[1 + \exp\left(\frac{ie\Phi}{\hbar} \right) \right]
\end{aligned}$$

Here, $\Phi = Ba$ is the magnetic flux enclosed by the two paths L and R, i.e., by the gold ring with enclosed area $a \simeq \pi d^2/4$. Hence, the influence of the magnetic field on the transmission probability is expressed as

$$T = |\psi_T|^2 \sim 1 + \cos\left(\frac{e\Phi}{\hbar} \right) = 1 + \cos\left(\frac{2\pi B}{B_0} \right),$$

with $B = \Phi/a$ and $B_0 = h/ea$. In other words, the conductance G, which is proportional with T, will oscillate with period h/ea as a function of the applied magnetic field B. The resistance $R = 1/G$ will of course oscillate with the same period.

b) The peak at $1/B_0 = 131\ \mathrm{T}^{-1}$ corresponds to $B_0 = 7.6$ mT, an enclosed area $a = h/eB_0 = 5.4 \cdot 10^{-13}\ \mathrm{m}^2 = 5.4 \cdot 10^5\ \mathrm{nm}^2$, and a ring diameter $d = \sqrt{4a/\pi} = 829$ nm. This is consistent with the size of the ring.

c) The quantum interference effect discussed above does not depend on the presence of a nonzero magnetic field B anywhere along the path taken by the electron, but only a nonzero enclosed magnetic flux. Therefore, this may be regarded as a nonlocal effect of the B-field. Gauge invariance means that measurable quantities depend on the *fields* and not (explicitly) on the *potentials*. The Aharonov - Bohm effect is gauge invariant, since $|\psi|^2$ depends on $\nabla \times \boldsymbol{A} = \boldsymbol{B}$.

d) Leonard (L. Hofstadter) is the experimental physicist in The Big Bang Theory.

QUESTION 4

a) A positive bias voltage V will tend to push electrons clockwise through the circuit, i.e., try to create a current I going counterclockwise. Assume we start at A with $U = U_0 - U_1$. The equilibrium situation now corresponds to a net charge of 2 extra electrons on the island. Adding or removing an electron is energetically unfavored, and electron tunneling is suppressed. By increasing U, we leave the $n = 2$ stability region and arrive at B where $U = U_0 + U_1$. Here, equilibrium corresponds to 3 extra electrons on the island, which is obtained by 1 electron tunneling onto the island through junction 1. On the next half period of $U(t)$, we move from B to A, where, again, $n = 2$ is the equilibrium situation. This is achieved by 1 electron tunneling out of the island through junction 2. During a full period of $U(t)$, the net effect is the transport of a single electron through the island. Hence, the current in the circuit is

$$I = \frac{\Delta Q}{\Delta t} = \frac{e}{T} = ef.$$

b) In order to clearly observe single electron charging effects, the thermal energy should be small compared to the electrostatic charging energy caused by the addition or removal of one electron. In other words,

$$T < \frac{e^2}{2k_B C}.$$

If $C = 1$ fF, the temperature should be (significantly) smaller than 1 K.

QUESTION 5

a) With only a single Landau level below E_F in the bulk region of the 2DEG, the transport of electrons take place via a single edge state, to the right along the upper edge and to the left along the lower edge. The direct transmission sums are

$$T_{21} = T_{32} = T_{43} = T_{14} = 1,$$

and all the others are zero. Putting this into the Büttiker–Landauer equations yields

$$\begin{aligned} I_1 &= \frac{2e^2}{h}(V_1 - V_4) \\ I_2 &= \frac{2e^2}{h}(V_2 - V_1) \\ I_3 &= \frac{2e^2}{h}(V_3 - V_2) \end{aligned}$$

$$I_4 = \frac{2e^2}{h}(V_4 - V_1)$$

With the given definition of the matrix $\boldsymbol{\gamma}$, we have

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

b) With terminals 1 and 3 as source and drain, respectively, the applied bias is $V_1 - V_3$, and we may, e.g., choose $V_3 = 0$. If terminals 2 and 4 are ideal voltage probes, we have immediately $I_2 = I_4 = 0$, which, by Kirchhoff's current rule implies $I_3 = -I_1$. The set of equations then reads

$$\begin{aligned} I_1 &= \frac{2e^2}{h}(V_1 - V_4), \\ 0 &= \frac{2e^2}{h}(V_2 - V_1), \\ -I_1 &= \frac{2e^2}{h}(V_3 - V_2), \\ 0 &= \frac{2e^2}{h}(V_4 - V_3), \end{aligned}$$

and we have, from the 2. equation, $V_2 = V_1$ and, from the 4. equation, $V_4 = V_3 = 0$. Thus, the Hall resistance is

$$R_H = \frac{V_2 - V_4}{I_1} = \frac{V_1}{I_1} = \frac{h}{2e^2},$$

and the 2-terminal resistance is

$$R_{2t} = \frac{V_1 - V_3}{I_1} = \frac{V_1}{I_1} = \frac{h}{2e^2}.$$

c) Interchange of the roles of terminals 2 and 3, i.e., with terminal 2 as drain and terminal 3 as voltage probe, we have $I_3 = 0$ and $I_2 = -I_1$. The equations are now

$$\begin{aligned} I_1 &= \frac{2e^2}{h}(V_1 - V_4), \\ -I_1 &= \frac{2e^2}{h}(V_2 - V_1), \\ 0 &= \frac{2e^2}{h}(V_3 - V_2), \\ 0 &= \frac{2e^2}{h}(V_4 - V_3). \end{aligned}$$

Here, we choose $V_2 = 0$ and obtain $V_2 = V_3 = V_4 = 0$. The 2-terminal resistance is

$$R_{2t} = \frac{V_1 - V_2}{I_1} = \frac{V_1}{I_1} = \frac{h}{2e^2},$$

as in **b)**. The longitudinal resistance is

$$R_L = \frac{V_3 - V_4}{I_1} = 0.$$

d) When the magnetic field strength is reduced, we also reduce the separation $\hbar\omega_c = \hbar eB/m^*$ between the Landau levels in the bulk region of the 2DEG. Each time an additional Landau level falls below the Fermi energy E_F, a new edge state becomes available for electron transport, and the nonzero elements of the matrix $\boldsymbol{\gamma}$ increase by one (in absolute value). As a consequence, the Hall resistance R_H drops from one plateau to another, from the initial value $h/2e^2$ when we have one edge state at E_F, to h/e^2, $3h/2e^2$, and so on. The transition between subsequent Hall plateaus is more or less "smooth", depending on the probability of scattering from one side of the sample to the other when a bulk Landau level lines up with the Fermi energy.
The longitudinal resistance remains zero, or close to zero, for most values of B. However, an increased probability of "back scattering" when a bulk Landau level lines up with the Fermi energy yields a nonzero R_L, and we observe so called Shubnikov - de Haas oscillations.