Problem 1: Weak Localization

(a) What is the origin of the weak localization effect ?

Solution

Weak localization arises from constructive quantum interference in a disordered solid. This gives rise to a quantum mechanical correction to the classical theory of conduction, the Drude formula. The origin is the enhanced quantum mechanical probability for an electron to return to its initial position. This is because a particle can return to its origin by following a clock-wise or counter-clock-wise loop. The quantum mechanical phases acquired along these two loops are identical. Therefore, the quantum mechanical probability for returning is twice the classical probability for return. As a consequence, weak localization enhances the resistivity and makes the system slightly more localized.

(b) Why is the effect called "weak" localization ?

Solution

The effects is a small correction to the resistivity that is a precursor to complete, or strong [Anderson], localization. It is therefore called "weak" localization.

(c) Why will a magnetic field affect the weak localization effect ?

Solution

Weak localization is caused by an increased probability for an electron to be back scattered. This is because a particle that returns to its origin can originate in two time-reversed paths that have exactly the same quantum mechanical phases. The quantum mechanical probability of returning is therefore twice the classical probability of returning. A magnetic field breaks time reversal symmetry, induces a phase difference between the two time-reversed paths, reduces the return probability, and enhances the conductivity.

Problem 2: The Quantum Hall Effect

Consider a two-dimensional electron gas in the x-y plane, where there is a transverse harmonic potential of the form $V(y) = m\omega_0^2 y^2/2$ and a magnetic field $\mathbf{B} = B\mathbf{z}$ applied along the z-direction. The Schrödinger equation is

$$\left[-\frac{\hbar^2}{2m}\left(\nabla + \frac{ie}{\hbar}\mathbf{A}\right)^2 + \frac{1}{2}m\omega_0^2 y^2\right]\psi(x,y) = E\psi(x,y)\,. \tag{1}$$

Choose the Landau gauge where the electromagnetic vector potential is $\mathbf{A} = (-yB, 0, 0)$.

(a) Use the Schrödinger equation (1) and the ansatz $\psi_{k,n}(x,y) = \phi_n(y) \exp ikx$ to show that the equation for ϕ_n is

$$\left[-\frac{d^2}{du^2} + (u-K)^2 + R^2u^2\right]\phi_n(u) = \epsilon_{k,n}\phi_n(u)\,, \tag{2}$$

where we have introduced the dimensionless variables $u = y/l_b$, $R = \omega_0/\omega_c$, $\epsilon = E/(\hbar\omega_c/2)$, $K = kl_B$. Additionally, $l_B = (\hbar/(eB))^{1/2}$ is the magnetic length, and $\omega_c = eB/m$ is the cyclotron frequency.

Solution

By using the Landau gauge, the Schrödinger equation (1) can be written as

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x} - \frac{iey}{\hbar}B\right)^2 - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}m\omega_0^2y^2\right]\psi(x,y) = E\psi(x,y)\,. \tag{3}$$

With $\psi_{k,n}(x,y) = \phi_n(y)\exp ikx$, $\partial/\partial x \to ik$ so that

$$\left(\frac{\partial}{\partial x} - \frac{iey}{\hbar}B\right)^2 \to \left(ik - \frac{iey}{\hbar}B\right)^2 = -\left(k - \frac{ey}{\hbar}B\right)^2\,. \tag{4}$$

We also make use of

$$\frac{\hbar^2}{2m}\frac{1}{l_B^2} = \frac{1}{2}\hbar\frac{eB}{m} = \frac{1}{2}\hbar\omega_c\,. \tag{5}$$

The first term in the Hamiltonian appearing in Eq. 3 is then

$$\frac{\hbar^2}{2m}\left(k - \frac{ey}{\hbar}B\right)^2 = \frac{\hbar^2}{2m}\frac{1}{l_B^2}\left(kl_B - \frac{y}{l_B}\right)^2 = \frac{1}{2}\hbar\omega_c\left(kl_B - \frac{y}{l_B}\right)^2\,. \tag{6}$$

The second term is

$$-\frac{\hbar^2}{2m}\frac{1}{l_B^2}\frac{\partial^2}{\partial(y/l_B)^2} = \frac{1}{2}\hbar\omega_c\frac{\partial^2}{\partial(y/l_B)^2} \tag{7}$$

and the third term is

$$\frac{1}{2}m\omega_0^2y^2 = \frac{1}{2}m\omega_c^2l_b^2\left(\frac{y}{l_B}\right)^2\frac{\omega_0^2}{\omega_c^2} = \frac{1}{2}\hbar\omega_c\left(\frac{y}{l_B}\right)^2\frac{\omega_0^2}{\omega_c^2}\,. \tag{8}$$

With $y = l_Bu$ and $k = K/l_B$ we find Eq. 2 with $\epsilon = E/(\hbar\omega_c/2)$, qed.

(b) Demonstrate that Eq. 2 can be re-written into the form of an equation for a particle in a harmonic potential and that the solution in terms of the original variables is

$$\psi_{k,n}(x,y) = e^{ikx}\phi_n(y - L_B^2k)\,, \tag{9}$$

where $\phi_n(y)$ is a harmonic oscillator function that satisfies

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}m\Omega^2y^2\right)\phi_n(y) = \hbar\Omega\left(n + \frac{1}{2}\right)\phi_n(y)\,, \tag{10}$$

$$L_B^2 = \frac{\omega_c^2}{\omega_c^2 + \omega_0^2} l_B^2 \tag{11}$$

and the eigenenergy for the eigenstate $\psi_{k,n}(x, y)$ is

$$E_{k,n} = \hbar\Omega(n + \frac{1}{2}) + \frac{\hbar^2 k^2}{2M_B} , \tag{12}$$

where $\Omega = (\omega_c^2 + \omega_0^2)^{1/2}$, and $M_B = m\Omega^2/\omega_0^2$.

Solution

The potential energy increases at most as a quadratic function in the transverse coordinate u, so we can re-write the potential terms as

$$(u - K)^2 + R^2 u^2 = \left(1 + R^2\right) \left(u - \frac{K}{1 + R^2}\right)^2 + K^2 - \frac{K^2}{1 + R^2} \tag{13}$$

This corresponds to a displaced harmonic oscillator potential term (quadratic in coordinate) with a re-defined energy:

$$\left[-\frac{d^2}{du^2} + (1 + R^2)(u - \frac{K}{1 + R^2})^2\right] \phi_n(u) = \left(\epsilon_{k,n} - \frac{R^2 K^2}{1 + R^2}\right) \phi_n(u) \tag{14}$$

Let us now return to the original variables:

$$1 + R^2 = 1 + \frac{\omega_0^2}{\omega_c^2} = \frac{\Omega^2}{\omega_c^2} , \tag{15}$$

$$u - \frac{K}{1 + R^2} = \frac{y}{l_B} - \frac{k l_B}{\Omega^2}\omega_c^2 = \frac{1}{l_B}\left(y - \frac{\omega_c^2}{\omega_0^2 + \omega_c^2} k l_B^2\right) = \frac{1}{l_B}\left(y - L_B^2 k\right) , \tag{16}$$

$$\frac{1}{2}\hbar\omega_c \left(\epsilon - \frac{\omega_0^2}{\omega_c^2}\frac{\omega_c^2}{\Omega^2} l_B^{"} k^2\right) = E - \frac{1}{2}\hbar\frac{eB}{m}\frac{\hbar}{eB}\frac{\omega_0^2}{\omega_0^2 + \omega_c^2} k^2 = E - \frac{\hbar^2}{2m}\frac{\omega_0^2}{\omega_0^2 + \omega_c^2} k^2 = E - \frac{\hbar^2 k^2}{2M_B} . \tag{17}$$

This implies that the energy levels are

$$E_{k,n} = \hbar\Omega\left(n + \frac{1}{2}\right) + \frac{\hbar^2 k^2}{2M_B} , \tag{18}$$

where $n = 0, 1, 2, \cdots$ and k is a continuum number. The harmonic oscillator functions are displaced and have arguments $y - L_B^2 k$.

(c) Compute the particle current density

$$\mathbf{j} = \frac{\hbar}{m}\mathrm{Im}\left(\psi^\dagger \nabla\psi\right) + \frac{e}{m}\mathbf{A}|\psi|^2 \tag{19}$$

for the eigenstate $\psi_{k,n}$ that we found above. What is the sign of the current density along the x-direction, j_x ?

Solution

Let us first consider the current along the y-direction. Since $A_y = 0$,

$$\mathrm{Im}\left(e^{-ikx}\phi_n(y - L_B^2 k)\frac{\partial}{\partial y}e^{ikx}\phi_n(y - L_B^2 k)\right) = 0\,, \tag{20}$$

and because the harmonic oscillator functions are real, we find that $j_y = 0$.

Second, we consider the current along the x-direction. In this case we have $A_x = -yB$ and we make use of

$$\mathrm{Im}\left(e^{-ikx}\phi_n\frac{\partial}{\partial x}e^{ikx}\phi_n\right) = k\phi_n^2 \tag{21}$$

so that

$$j_x = \left(\frac{\hbar k}{m} - \frac{eyB}{m}\right)\phi_n^2(y) = -\omega_c\left(y - l_B^2 k\right)\phi_n^2(y - L_B^2 k) \tag{22}$$

Since $\phi_n^2(y - L_B^2 k)$ is always positive, $j_x \leq 0$ when $y > l_B^2 k$ and $j_x \geq 0$ when $y < l_B^2 k$ irrespective of the quantum number n and the frequencies ω_c and ω_0.

Problem 3: The Landauer-Büttiker formalism

The Landauer-Büttiker formula for the conductance is

$$G = \frac{e^2}{h}\sum_n T_n\,, \tag{23}$$

where T_n is the transmission probability for transverse wave guide mode n and the sum is over transverse wave guide modes.

(a) Consider a one-dimensional system (only one wave guide mode) with a left and right reservoir with chemical potentials μ_L and μ_R, respectively, such that $eV = \mu_L - \mu_R$. Find arguments for how the current should be expressed in terms of the velocity $v(\epsilon)$ of an electron at energy ϵ, the density of states in one dimension $N(\epsilon) = 2/[hv(\epsilon)]$, the transmission probability $T(\epsilon)$, and the distribution functions in the left and the right reservoirs $f(\epsilon - \mu_L)$ and $f(\epsilon - \mu_R)$. Derive from these arguments the Landauer-Büttiker conductance in the linear response regime (the bias voltage is much smaller than the Fermi energy) at zero temperature, Eq. 23 with only one transverse wave guide mode, $G = (e^2/h)T$.

Solution

The probability of finding an electron at energy ϵ in the left (right) reservoir is determined by the Fermi-Dirac distributiuon function $f(\epsilon - \mu_L)$ $(f(\epsilon - \mu_R))$.

The current consists of right-going and lef-going particles. The probability that an electron will move from the left reservoir to the right reservoir is

$$P_{l\to r}(\epsilon) = f(\epsilon - \mu_l)\,[1 - f(\epsilon - \mu_R)]\,T(\epsilon)\,. \tag{24}$$

Similarly, the probability that an electron will move from the right reservoir to the right reservoir is

$$P_{r\to l}(\epsilon) = f(\epsilon - \mu_r)\,[1 - f(\epsilon - \mu_l)]\,T(\epsilon)\,. \tag{25}$$

The net current produced by electrons with energy ϵ is

$$I(\epsilon) = N(\epsilon) e v(\epsilon)\,[P_{l\to r} - P_{r\to l}]\ . \tag{26}$$

where $N(\epsilon) = 2/[hv(\epsilon)]$ is the density of states in one dimension. The total curent is then

$$I = \frac{2e}{h}\int d\epsilon\,(f(\epsilon-\mu_r) - f(\epsilon-\mu_l))\,T(\epsilon) \tag{27}$$

We consider the linear response regime where $eV = \mu_l - \mu_r$ is small. We may then expand

$$f(\epsilon-\mu_l) \approx f(\epsilon-\mu_0) + (\mu_l-\mu_0)\left(-\frac{\partial f(\epsilon-\mu_0)}{\partial\epsilon}\right) \tag{28}$$

and

$$-f(\epsilon-\mu_r) \approx f(\epsilon-\mu_0) + (\mu_r-\mu_0)\left(-\frac{\partial f(\epsilon-\mu_0)}{\partial\epsilon}\right) \tag{29}$$

where μ_0 is the equilibrium chemical potential.

At low temperatures

$$\left(-\frac{\partial f(\epsilon-\mu_0)}{\partial\epsilon}\right) = \delta(\epsilon-\mu_0) \tag{30}$$

so that the total current becomes

$$I = \frac{2e^2}{h}T(\mu_0)V \tag{31}$$

and the conductance $G = I/V$ is

$$G = \frac{2e^2}{h}T(\mu_0) \tag{32}$$

q.e.d.

(b) We consider a narrow quantum wire that is created in a two-dimensional electron gas such that the system is infinite in the x-direction, but has a finite width L in the y-direction. The potential is assumed to be infinite outside the wire where the wave function vanishes. We assume there is no scattering in the wire and transport is ballistic. Show that the conductance is

$$G = \frac{2e^2}{h}\left[\frac{Lk_F}{\pi}\right], \tag{33}$$

where $[\cdots]$ represents the integral part of the number, e.g. $[2.1] = 2$.

Solution The solution can be found as follows. The energy of the system has three contributions arising from the motion along the x and y directions. The particle is free to move along x and the energy associated with this motion is E_x. Along the transverse direction y, the wave function must represent standing waves of the form $\psi(y) = A \sin n\pi y/L$, where A is a normalization constants and $n = 0, 1, 2, 3, \cdots$. The total energy of the system is then

$$E = E_x + \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2 . \tag{34}$$

The wave guide modes are only propagating when $E_x > 0$. Transport is governed by electrons at the Fermi energy $E_F = \hbar^2 k_F^2/(2m)$. This implies that the propagating modes are determined by

$$E_F \geq \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2 \tag{35}$$

so that

$$n \leq \left(\frac{Lk_F}{\pi}\right) . \tag{36}$$

Since transport is ballistic, the transmission probability $T_n = 1$ for all propagating modes. Keeping in mind that the quantum number n is an integral number, the Landauer-Buttiker condutance is therefore

$$G = \frac{2e^2}{h}\left[\frac{Lk_F}{\pi}\right] . \tag{37}$$

Problem 4: Magnetoresistance

(a) What do the abbrevations AMR, GMR, and TMR mean ? What is the cause of AMR, GMR, and TMR ?

Solution

AMR - Anisotropic magnetoresistance. GMR - giant magnetoresistance. TMR - tunnel magnetoresistance.

When a current passes through a ferromagnet, the resistance depends on the relative direction of the current and the magnetization. This anistropic magnetoresistance is caused by the spin-orbit coupling.

Giant magnetoresistance occurs in metallic hybrid systems of ferromagnets and normal metals. The resistance depends on the relative orientation of two or more ferromagnets and is caused by spin-dependent scattering in the ferromagnets. Electrons aligned or anti-aligned to the magnetization experience different potentials and have different conductances.

Tunnel magnetoresistance occurs in transport through tunnel junctions between ferromagnets. The current depends on the relative orientation of the ferromagnerts and is caused by spin-dependent tunneling.