Problem 1: Various

(a) Define the following length scales that are relevant in discussing transport in nano-scale systems: 1) Fermi wavelength λ_F, 2) elastic mean free path l_e, and 3) phase coherence length l_ϕ.

Solution

1) The Fermi wavelength is the wavelength of the electrons at the Fermi energy. In a free electron gas, it is defined via $E_F = \hbar^2 k_F^2/2m$, where E_F is the Fermi energy, $k_F = 2\pi/\lambda_F$ is the Fermi wavevector in terms of the Fermi wavelength λ_F and m is the electron (effective) mass. In linear response, the net current is mostly carried by electrons at the Fermi energy, so the typical electron wavelength for transport electrons is the Fermi wavelength.

2) The elastic mean free path is the average distance an electron travels between elastic collisions (collisions where the electron's momentum changes but not its energy).

3) The phase coherence length is the average distance traveled while the electron maintains a well-defined quantum mechanical phase. The quantum mechanical phase is randomized by inelastic scattering arising e.g. from electron-electron or electron-phonon scattering.

(b) We consider a 1D channel and define a scattering matrix consisting of reflection r (r') and transmission amplitudes t (t') for electrons coming from the left (right) lead as

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} . \tag{1}$$

Imagine that we have two scatterers in series with scattering matrices S_1 and S_2, respectively.

Make the assumption that there is incoherent scattering in the 1D channel so scattering by the two potentials is incoherent. What is the total transmission probability through the system in terms of the elements of the scattering matrices S_1 and S_2 in this case?

Solution

There are two incoherent scatterers in the system with transmission (reflection) probabilities $T_1 = |t_1|^2$ ($R_1 = |r_1|^2$) and $T_2 = |t_2|^2$ ($R_2 = |r_2|^2$), respectively. Since the system is incoherent, we should now sum the probabilities, and not the amplitudes, for all possible trajectories an electron can take when it is transmitted through the system. The total (incoherent) transmission probability is then

$$T_{12}^{(i)} = T_1T_2 + T_1R_2R_1T_2 + T_1(R_2R_1)^2T_2 + \ldots \tag{2}$$

$$= \frac{T_1T_2}{1 - R_2R_1} = \frac{1}{T_1^{-1} + T_2^{-1} - 1}\,. \tag{3}$$

Problem 2: The Landauer-Büttiker formula

The Landauer-Büttiker formula for the relation between currents I_i and voltages V_j in a many-terminal system reads

$$I_i = \sum_j \left[G_{ji}V_i - G_{ij}V_j\right], \tag{4}$$

where the sum is over the terminal indices $j = 1, 2, 3, \ldots N$ and $i = 1, 2, 3$ etc., N is the number of terminals, and G_{ij} are elements of a conductance matrix.

(a) Prove that $\sum_j G_{ji} = \sum_j G_{ij}$.

Solution

In equilibrium, all potentials are equal, $V_i = V$, and all currents must vanish, $I_i = 0$. This requires that

$$0 = V \sum_j \left[G_{ji} - G_{ij}\right] \tag{5}$$

for any voltage V so that

$$\sum_j G_{ji} = \sum_j G_{ij}\,. \tag{6}$$

(b) In the remainder of the problems, we will assume that $G_{ij} = G_{ji}$. Which physical condition must be fulfilled to justify this assumption?

Solution

This assumption is fulfilled when there is time-reversal symmetry, e.g. no orbital magnetic field effects so that there is an equal probability to go from terminal i to terminal j as vice versa.

(c) Consider a three-terminal device, where a current $I = I_1 = -I_2$ passes from terminal 1 at voltage V_1 to terminal 2 at voltage V_2. In response to the current I, there is a voltage V_3 at the terminal 3 where there is no current, $I_3 = 0$.

Consider first (and here only) that the conductance element G_{23} is much bigger than all the other elements of the conductance matrix. In this limit, what is the voltage difference $V_3 - V_2$?

Solution When the conductance G_{23} is much bigger than all the other elements of the conductance matrix, it implies that there is a strong electrical connection between terminals 2 and 3 so that the voltage difference $V_3 - V_2$ will approach zero.

(d) Compute the potential difference $V_3 - V_2$ as a function of the current I and the conductance matrix G_{ij} of the system.

Solution

We have three equations for the three currents in the system:

$$I_1 = G_{12}(V_1 - V_2) + G_{13}(V_1 - V_3) \tag{7}$$

$$I_2 = G_{12}(V_2 - V_1) + G_{23}(V_2 - V_3) \tag{8}$$

$$I_3 = G_{23}(V_3 - V_2) + G_{13}(V_3 - V_1) \tag{9}$$

where we have used that $G_{ij} = G_{ji}$.

Only the relative voltage differences matter and we may choose $V_2 = 0$ so that V_3 represents the voltage difference $V_3 - V_2$. Since there is no current flowing into terminal 3, $I_3 = 0$ determines that

$$V_3 = \frac{G_{13}}{G_{23} + G_{13}} V_1 \,. \tag{10}$$

We then find that

$$V_1 - V_3 = \frac{G_{23}}{G_{23} + G_{13}} V_1 \,. \tag{11}$$

The current $I = I_1$ is then

$$I = \left(G_{12} + \frac{G_{13}G_{23}}{G_{23} + G_{13}} \right) V_1 \tag{12}$$

$$= \frac{G_{12}\,(G_{23} + G_{13}) + G_{13}G_{23}}{G_{13}} V_3 \tag{13}$$

so that the potential is

$$V_3 - V_2 = \frac{G_{13}}{G_{12}\,(G_{23} + G_{13}) + G_{13}G_{23}} I \,. \tag{14}$$

In the limit that $G_{23} \gg G_{12}$ and $G_{23} \gg G_{13}$, we find that $V_3 \to V_2$ because the terminals 2 and 3 are strongly coupled, as we found in the previous problem.

Problem 3: The Quantum Hall effect

(a) Give a physical explanation of what Landau levels are.

Solution

Classically, electrons in a two-dimensional electron gas form cyclotron orbits in the presence of a magnetic field in a perpendicular direction. Landau levels are the quantization of these classical cyclotron orbits. The energy separation is determined by the classical cyclotron frequency multiplied by Planck's constant.

(b) Give a physical explanation of what semi-classical skipping orbits is as well as what their quantum mechanical analogue are.

Solution

As stated above, electrons in a two-dimensional film in the presence of a perpendicular magnetic field form cyclotron orbits. These orbits do not acquire a drift velocity in the presence of an electric field. However, at the edges of the film, skipping orbits are formed by the bouncing off of the electrons from the interface. These skipping orbits lead to a net motion of the electrons along the wall, in opposite directions on opposite walls. The quantum mechanical analogues are edge states, quantum confined states close to the interface can carry currents while the bulk of the system is insulating.

Problem 4: Spintronics

(a) Explain what the following spintronics phenomena are: i) giant magnetoresistance (GMR) and 2) spin-transfer torques.

Solution

1) Giant magnetoresistance occurs in layered normal metal-ferromagnet systems and was first seen in a superlattice made by a repetition of this system. For specific thicknesses of the normal metal layers, the ferromagnets align in an anti-parallel configuration like a synthetic macroscopic anti-ferromagnet. The relative orientation of the magnetizations of adjacent ferromagnets sandwiched via normal metals can be changed by applying an external magnetic field. As a result, a change of resistance also follows. This is a consequence of spin-dependent transport in the ferromagnets. The conductances differ for spin-up and spin-down electrons. When the ferromagnets are parallel, the conductances associated with one spin-channel is large and the other small,

whereas for the anti-parallel orientation both spin-channels have a relatively small conductance. This causes the conductance for the anti-parallel configuration to be considerably smaller than the one for the parallel configuration, which gives rise to a magneto resistance. This magneto resistance is called "giant" since it was much bigger than other known effects (in 1988). The resistance change was roughly a factor of 2.

2) When an electron flows between two magnetic domains (regions with uniform magnetization direction) its spin direction changes because the strong exchange interaction forces the electron spin to align to the magnetization direction of the new domain. Even when driven by an electric field, the magnetic moments of majority and minority spins remain parallel and antiparallel to the magnetization, respectively. By conservation of spin angular momentum, the change of the electron spin on entering the new domain yields a small change of the magnetization direction of this domain, which can be viewed as arising from a torque. This central concept of spin-transfer torque (STT) requires non-collinear magnetization configurations and has been the focus of spintronics research for the past fifteen years.

(b) We consider the Pauli equation for spin 1/2 electrons in a one-dimensional channel along the x-direction and disregard the spin-orbit interaction. We assume that there is a weak magnetic field B applied along the z-direction confined to a length $-L/2 < x < L/2$ and include its effect on the spin degrees of freedom only. The Hamiltonian is then

$$H = \begin{cases} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}I + \frac{ge\hbar}{2m}B\sigma_z & ; -L/2 < x < L/2 \\ -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}I & ; |x| \geq L/2 \end{cases} , \tag{15}$$

where the unit matrix I and the Pauli-matrix σ_z are defined as

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{16}$$

and

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{17}$$

We may define a Zeeman energy

$$E_Z = \frac{ge\hbar}{2m}B \tag{18}$$

and assume that $E_Z > 0$ throughout this exam.

Consider an electron with energy $0 < E < E_Z$. Assume (in this question only) that the system is much longer than any other possible length scale, $L \to \infty$. Argue, without explicit calculations, what the reflection probability $R_\uparrow$ is for an incoming spin-up electron. For your information, the spin-up electron sees a larger potential barrier than the spin-down electron in the barrier region $-L/2 < x < L/2$.

Solution

For a spin-up electron, the energy of the electron is less than the Zeeman energy that acts as a barrier. Since the system is infinitely long, the electron will then not propagate through the system and will be completely reflected, $R_\uparrow = 1$.

(c) Consider still an electron with energy $0 < E < E_Z$, but the barrier region has now a finite length L. Compute the transmission probability for the passage of an electron through the one-dimensional channel for both a spin-up state and a spin-down state.

Solution

In the leads, when $|x| > L/2$, the wave function is $\psi(x) \sim \exp \pm ikx$ for electrons propagating to the right (+) and left (−). The eigenenergy is $E = \hbar^2 k^2/2m$, where k is the wave vector.

Inside the channel, we must treat the evolution of the amplitude of the spin-up and spin-down electrons differently since only the spin-down electrons can propagate.

We define the wavevector of the spin-down electron inside the system as $q_\downarrow = \sqrt{2m(E + E_Z)}/\hbar$ and similarly the decaying length scale for spin-up electrons is defined as $q_\downarrow = \sqrt{2m(E_Z - E)}/\hbar$.

We consider an incoming electron with spin-down from the left. The wave function can then be expressed as

$$\psi_\downarrow(x) = \begin{cases} \exp ik(x + L/2) + r_\downarrow \exp -ik(x + L/2) & x \leq -L/2 \\ A \exp iq_\downarrow x + B \exp -iq_\downarrow x & -L/2 < x < L/2 \\ t_\downarrow \exp ik(x - L/2) & x \geq L/2 \end{cases} \tag{19}$$

Continuity of the wave function and its derivative determine at $x = -L/2$ and $x = L/2$ that

$$1 + r_\downarrow = A \exp -iq_\downarrow L/2 + B \exp +iq_\downarrow L/2\,, \tag{20}$$

$$ik(1 - r_\downarrow) = iq_\downarrow \left[A \exp -iq_\downarrow L/2 - B \exp +iq_\downarrow L/2\right]\,, \tag{21}$$

$$t_\downarrow = A \exp iq_\downarrow L/2 + B \exp -iq_\downarrow L/2\,, \tag{22}$$

$$ikt_\downarrow = iq_\downarrow \left[A \exp iq_\downarrow L/2 - B \exp -iq_\downarrow L/2\right] \,. \tag{23}$$

Dividing Eq. (21) by ik and adding Eq. (20) gives

$$2 = A\frac{k+q_\downarrow}{k} \exp -iq_\downarrow L/2 + B\frac{k-q_\downarrow}{k} \exp +iq_\downarrow L/2 \tag{24}$$

Similarly, dividing Eq. (23) by ik and subtracting Eq. (23) from Eq. (22) gives

$$0 = A\frac{k-q_\downarrow}{k} \exp iq_\downarrow L/2 + B\frac{k+q_\downarrow}{k} \exp -iq_\downarrow L/2 \,. \tag{25}$$

We then find that

$$A = -B\frac{k+q_\downarrow}{k-q_\downarrow} \exp -iq_\downarrow L \,. \tag{26}$$

Inserting this result into Eq. (24) gives

$$B = \frac{2k(k-q_\downarrow)}{(k-q_\downarrow)^2 \exp i2q_\downarrow L - (k+q_\downarrow)^2} \exp i(3/2)q_\downarrow L \tag{27}$$

$$A = -\frac{2k(k+q_\downarrow)}{(k-q_\downarrow)^2 \exp i2q_\downarrow L - (k+q_\downarrow)^2} \exp iq_\downarrow L/2 \tag{28}$$

From this, we can finally find the transmission coefficient from Eq. (23):

$$t_\downarrow = \frac{2ikq_\downarrow}{2ikq_\downarrow \cos Lq + (k^2 + q_\downarrow^2) \sin Lq} \,. \tag{29}$$

The transmission probability is then

$$|t_\downarrow|^2 = \frac{4k^2 q_\downarrow^2}{4k^2 q_\downarrow^2 \cos^2 Lq + (k^2 + q_\downarrow^2)^2 \sin^2 Lq} \,. \tag{30}$$

As a check, consider the limit of no Zeeman splitting, $q_\downarrow \to k$, where we find

$$t_\downarrow \to \exp ikx \tag{31}$$

which corresponds to the phase acquired as the electron passes through the system so that, in this case, $|t_\downarrow|^2 \to 1$.

The result for the transmission coefficient of Eq. (29) can be directly carried over to the spin-up electron case with the substitution $q_\downarrow \to iq_\uparrow$. Since we then also have $\cos q_\downarrow L \to \cosh q_\uparrow L$ and $\sin q_\downarrow L \to i \sinh q_\uparrow$, we find

$$t_\uparrow = \frac{2ikq_\uparrow}{2kiq_\uparrow \cosh Lq + (k^2 - q_\uparrow^2) \sinh Lq} \tag{32}$$

so that

$$|t_\uparrow|^2 = \frac{4k^2 q_\uparrow^2}{4k^2 q_\uparrow^2 \cosh^2 Lq + (k^2 - q_\uparrow^2)^2 \sinh^2 Lq} \tag{33}$$

In the limit that $Lq \gg 1$, the transmission probability for the spin-up electron is then exponentially small, as expected since the energy of the spin-up electron is smaller than the Zeeman potential barrier.