Problem 1: Electron Concentration and Fermi Wavelength

(a) Calculate the Fermi wavevector k_F of a three dimensional electron gas (3DEG) expressed in terms of the electron concentration n at a temperature $T = 0$. Do the same for 2D and 1D systems. You should include the spin-degeneracy $g = 2$ in all these cases.

Solution

The Fermi wavevector k_F is defined in terms of the Fermi energy E_F via

$$E_F = \frac{\hbar^2 k_F^2}{2m} . \tag{1}$$

At zero temperature, all states are occupied up to the Fermi level. Using periodic boundary conditions, one finds that the separation between the states in the reciprocal lattice is $2\pi/L_i$ in each direction of length L_i.

In a 3DEG, the electron concentration is then

$$n_3 = g \frac{1}{(2\pi)^3} \frac{4\pi}{3} k_F^3 = \frac{k_F^3}{3\pi^2} , \tag{2}$$

where $g = 2$ is the spin-degeneracy factor.

The Fermi wavevector is therefore

$$k_F = (3\pi^2 n_3)^{1/3}. \tag{3}$$

In a 2DEG, the electron concentration is

$$n_2 = g \frac{1}{(2\pi)^2} \pi k_F^2 = \frac{k_F^2}{2\pi} . \tag{4}$$

The wavevector is therefore

$$k_F = (2\pi n_2)^{1/2} . \tag{5}$$

In a 1DEG, the electron concentration is

$$n_1 = g \frac{1}{(2\pi)} 2k_F = \frac{2k_F}{\pi} . \tag{6}$$

The wavevector is therefore

$$k_F = \pi n_1/2 . \tag{7}$$

(b) The dimensional unit in which the electron concentration is expressed changes with the dimensionality of the system. To compare systems with different dimensionalities, the electron concentration can be expressed in terms of the effective distance between the electrons d_{e-e} by assuming that each electron resides in a box of size d_{e-e}.

Express the results for k_F in one, two, and three dimensions in terms of d_{e-e} and calculate the numerical value of the prefactors.

Solution

The electron density can be expressed in terms of the box size as

$$n_3 = 1/d_{e-e}^3\,, \tag{8}$$

$$n_2 = 1/d_{e-e}^2\,, \tag{9}$$

$$n_1 = 1/d_{e-e}\,. \tag{10}$$

In 3D, the wavevector is therefore expressed in terms of the box size as

$$k_F = (3\pi^2)^{1/3}/d_{e-e} \approx 3.1/d_{e-e}\,. \tag{11}$$

Similarly, in 2D and 1D, we find

$$k_F = (2\pi)^{1/2}/d_{e-e} \approx 2.5/d_{e-e}\,, \tag{12}$$

and

$$k_F = (\pi/2)/d_{e-e} \approx 1.6/d_{e-e}\,, \tag{13}$$

respectively.

Problem 2: Weak Localization

(a) What is weak localization?

Solution

Weak localization arises from constructive quantum interference in a disordered solid. This gives rise to a quantum mechanical correction to the classical theory of conduction, the Drude formula. The origin is the enhanced quantum mechanical probability for an electron to return to its initial position. This is because a particle can return to its origin by following a clock-wise or counter-clock-wise loop. The quantum mechanical phases acquired along these two loops are identical. Therefore, the quantum mechanical probability for returning is twice the classical probability for return. As a consequence, weak localization enhances the resistivity and makes the system slightly more localized. The effect is a small correction to the resistivity that is a precursor to complete, or strong [Anderson], localization. It is therefore called weak localization.

(b) Sketch the resistance versus the magnetic field in a device in which weak localization occurs.

Solution

A magnetic field breaks the time-reversal symmetry so that quantum mechanical phases along the clock-wise and counter-clock-wise loops become different. This reduces the probability for return and decreases the resistance.

A sketch of the resistance versus magnetic field is therefore as in Fig. 1.

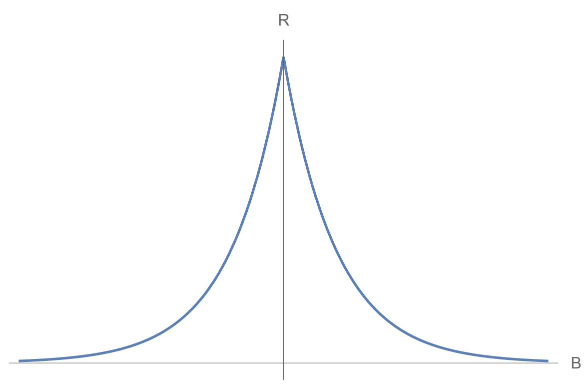

FIG. 1. A sketch of the resistance versus magnetic field illustrating the weak localization effect. The magnetic field breaks the time-reversal symmetry so that the return probability is reduced and the resistance decreases.

Problem 3: Coherent Transport

A one dimensional conductor contains a single scatterer with transmission coefficient t and an electron with wavefunction $\psi(x) = \exp ikx$ is sent into the conductor. The incoming, reflected, and transmitted waves on both sides of the scatterer as well as the left and right reservoirs are shown in Fig. 2.

(a) The current density J is given by

$$J = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{dx}\right) . \tag{14}$$

Express the current density in terms of t and the velocity of the electron.

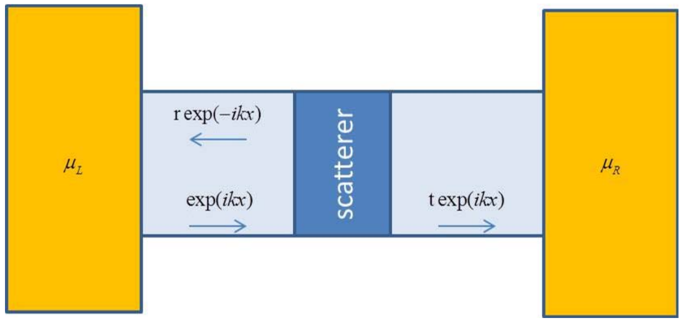

FIG. 2. A one dimensional conductor with a single scatterer connected to a left and a right reservoir. The wave function to the left of the scatterer is $\psi(x) = \exp ikx + r\exp -ikx$ and the wave function to the right of the scatterer is $\psi(x) = t\exp ikx$.

Solution

To the right of the scatterer, the probability current is

$$J = \frac{\hbar}{2mi}\left[t^* e^{-ikx}(ik)te^{ikx} - te^{ikx}(-ik)t^* e^{-ikx}\right] = v|t|^2 \,. \tag{15}$$

We can similarly compute the probability current to the left of the scatterer and will find the exact same result because of conservation of particle flow.

(b) Assume that the scatterer is a rectangular potential of height $V_0 > E_F$ and width d within $-d/2 < x < d/2$. Calculate the (energy-dependent) transmission probability $T(E) = |t(E)|^2$.

Solution

The wavefunction is

$$\psi(x) = \begin{cases} \exp ikx + r\exp -ikx & , \quad x < -d/2 \\ A\exp qx + B\exp -qx & , \quad -d/2 < x < d/2 \\ t\exp ikx & , \quad x > d/2 \end{cases} , \tag{16}$$

where the wavevector k and the inverse decay length q are defined via

$$\frac{\hbar^2 k^2}{2m} = E \tag{17}$$

$$\frac{\hbar^2 q^2}{2m} = V_0 - E \,. \tag{18}$$

Continuity of the wavefunction gives

$$\exp -ikd/2 + r\exp ikd/2 = A\exp -qd/2 + B\exp qd/2 \tag{19}$$

$$A\exp qd/2 + B\exp -qd/2 = t\exp ikd/2\,. \tag{20}$$

Continuity of the first derivative of the wave function gives

$$ik(\exp -ikd/2 - r\exp ikd/2) = q(A\exp -qd/2 - B\exp qd/2) \tag{21}$$

$$q(A\exp qd/2 + B\exp -qd/2) = ikt\exp ikd/2\,. \tag{22}$$

Solving these equations, we find

$$t = \frac{2ikqe^{-ikd}}{2ikq\cosh qd - (k^2-q^2)\sinh qd} \tag{23}$$

so that the transmission probability is

$$T = \frac{1}{\cosh^2 qd + \frac{(k^2-q^2)^2}{4k^2q^2}\sinh^2 qd}\,, \tag{24}$$

where the relation between k and q and the energy are defined above.

Problem 4: Single-electron Tunneling and Coulomb Blockade

(a) An amount of charge Q_0 is placed on a capacitor with capacitance C (there is a charge $+Q_0$ on one side of the capacitor and $-Q_0$ on the other side of the capacitor). There is a resistance R to the ground, see Fig. 3. Show that the charge $Q(t)$ on the capacitor evolves as

$$Q(t) = Q_0\exp -t/\tau \tag{25}$$

and determine the decay time τ.

Solution

Kirchhoff's second law, the voltage law, implies for this circuit that the voltage across the capacitance together with the voltage across the resistor must vanish. We then find

$$\frac{Q}{C} + \frac{dQ}{dt}R = 0 \tag{26}$$

so that

$$\frac{dQ}{dt} = -\frac{Q}{RC} \tag{27}$$

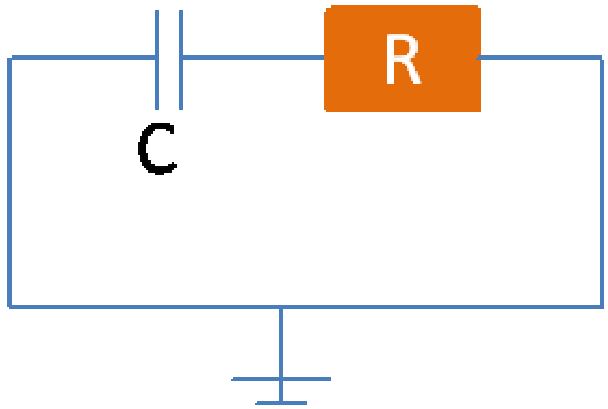

FIG. 3. An electric circuit with a capcitance in series with a resistor.

which has as a solution

$$Q(t) = Q_0 \exp -t/\tau\,, \tag{28}$$

where the decay time is

$$\tau = RC\,. \tag{29}$$

(b) The Heisenberg uncertainty principle states that the energy of an electron is ill defined when the electron stays in a state only for a short time:

$$\Delta E \Delta t \geq \hbar/2\,. \tag{30}$$

What is the uncertainty in energy for the system discussed in Problem 4a?

Solution

The lifetime for the system is the decay time τ. From the Heisenberg relation (30), we then find that the uncertainty in the energy is

$$\Delta E \geq \frac{\hbar}{2\tau} = \frac{\hbar}{2RC}\,. \tag{31}$$

(c) We now assume that we have a nano-scale system where a junction exhibits both capacitive effects and tunneling, e.g. that a single junction consists both of a capacitance C and a (tunnel) resistance R as in Fig. 3.

Find a condition, expressed in terms of the tunnel conductance $G = 1/R$, for when quantum fluctuations can be avoided so that single-electron tunneling effects can be clearly seen.

Solution

The classical charging energy associated with the tunneling of one electron is $E_c = e^2/2C$. From Heisenberg's uncertainty relation, we have

$$\Delta\epsilon\Delta t \geq \hbar/2\,, \tag{32}$$

where $\Delta\epsilon$ is the energy uncertainty and Δt is the lifetime uncertainty. Since we know that the decay time is $\tau = RC$, we find that

$$\Delta\epsilon \geq \frac{\hbar}{2\tau}\,. \tag{33}$$

In order to clearly see single-electron tunneling effects these energy fluctuations must be smaller than the typical charging energy associated with a single electron, $\Delta\epsilon \ll E_C$. This implies that

$$\frac{\hbar}{2\tau} \ll E_C\,. \tag{34}$$

Using $\tau = RC$ and $E_C = e^2/2C$ for a single electron, we also find

$$G \ll \frac{2e^2}{h}\pi\,. \tag{35}$$

In other words, the tunneling resistance must be large. Alternatively, the tunnel conductance must be much smaller than the conductance quantum. The sum of all transmission probabilities must be much smaller than 1.

Problem 5: Landau Levels

We consider electrons in a two-dimensional electron gas (2DEG) subject to a perpendicular magnetic field. We disregard the effects of the electron spin. The Schrödinger equation is then

$$\frac{(\mathbf{p}+e\mathbf{A})^2}{2m}\psi(x,y) = E\psi(x,y)\,, \tag{36}$$

where $-e$ is the electron charge and $\mathbf{A}$ is the electromagnetic vector potential.

(a) The perpendicular magnetic field is $\mathbf{B} = B\mathbf{z}$, where B is its magnitude and $\mathbf{z}$ is a unit vector in the z-direction. Let us assume that the electromagnetic vector potential only depends on the in-plane coordinates x and y and only has components along the x and

y directions. Find all possibilities for the vector potential $\mathbf{A}$ to the first order in the coordinates x and y, e.g. $\mathbf{A}$ is of the form

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1 x + \mathbf{A}_2 y\,. \tag{37}$$

when we use the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0\,. \tag{38}$$

Solution

The electromagnetic vector potential is $\mathbf{A} = (A_x(x,y), A_y(x,y), 0)$.

The magnetic field is determined by the relation

$$\mathbf{B} = B\mathbf{z} = \nabla \times \mathbf{A} = \mathbf{z}(\partial_x A_y - \partial_y A_x) \tag{39}$$

To linear order in the coordinates x and y, we can write

$$A_y = A_{0y} + A_{1y}x + A_{2y}y \tag{40}$$

$$A_x = A_{0x} + A_{1x}x + A_{2x}y \tag{41}$$

and we should therefore satisfy

$$(\partial_x A_y - \partial_y A_x) = B \tag{42}$$

$$A_{1y} - A_{2x} = B \tag{43}$$

With these contraints, the electromagnetic vector potential is therefore

$$A_y = A_{0y} + A_{1y}x + A_{2y}y \tag{44}$$

$$A_x = A_{0x} + A_{1x}x + (A_{1y} - B)y \tag{45}$$

where A_{0y}, A_{0x}, A_{1y}, A_{1x}, and A_{2y} are gauge-dependent coefficients. Next, we should satisfy the Coulomb gauge:

$$\nabla \cdot \mathbf{A} = 0 \tag{46}$$

$$A_{1x} + A_{2y} = 0\,. \tag{47}$$

This implies that the electromagnetic vector potential is

$$A_y = A_{0y} + A_{1y}x - A_{1x}y \tag{48}$$

$$A_x = A_{0x} + A_{1x}x + (A_{1y} - B)y\,. \tag{49}$$

(b) It can be shown that the eigenvalues of the Schrödinger equation (36) in the Landau gauge $\mathbf{A} = (0, Bx, 0)$ are

$$E_n = \hbar\omega_c \left(n + \frac{1}{2}\right) \tag{50}$$

where n is an integral number $n = 0, 1, 2, \ldots$ and $\omega_c = eB/m$ is the cyclotron frequency. Find the eigenvalues in the other possible choices of the gauge for the electromagnetic vector potential.

Solution

The eigenvalues of the Schrödinger equation are always gauge-invariant. The eigenvalues are therefore the same in all possible choices of the gauge although the wave function changes with the gauge.