Department of physics, NTNU
TFY4340 Mesoscopic Physics

Solutions exam problems

1. **Short qualitative questions. (12 points)**

   Use only a few sentences to answer each question.

   a. (4 points) Bloch states are *extended* states, given as a product of a plane wave and a function periodic in the lattice. Surface states are *localized* to the surface(s) of the system.

   b. (4 points) Weak localization manifests as a slight decrease of the conductivity, as compared to the classically predicted value. It results from constructive interference of time-reversed trajectories along closed loops, which contribute to the probability of backscattering for electrons propagating through a strongly disordered sample. An applied magnetic field breaks time-reversal symmetry and thus destroys the constructive interference, thereby leading to an increase in the conductivity.

      The field scale B_c at which the increase starts to become significant (on the order of the weak-localization correction itself) corresponds to a flux of order 1 penetrating the largest loops that contribute to the localization. The paths defining these loops are of length $\sim l_\varphi$ and thus $B_c l_\varphi^2 \sim 1$, which gives an estimate for l_φ for a given observed B_c.

   c. (4 points) The effective mass follows from

   $$\frac{1}{m^*} = \frac{1}{\hbar^2}\frac{d^2E}{dk^2}. \tag{1}$$

      Use $E = p^2/2m^* = \hbar^2k^2/2m^*$ to remember the details.

2. **Drude formula. (8 points)**

   We use Newton's second law,

   $$\frac{d\mathbf{p}}{dt} = \mathbf{F}. \tag{2}$$

   The force is in this case the electric field acting on an electron, $\mathbf{F} = -eE_x\hat{x}$, where we defined the x-direction as being parallel to the field (without any loss of generality). We thus have

   $$m^*\frac{dv_x}{dt} = -eE_x. \tag{3}$$

Scattering events randomize the direction of propagation of an electron, so after each collision the expectation value for the velocity of the electron is $\langle \mathbf{v} \rangle = 0$. After a collision, only v_x is time-dependent,

$$v_x = -\frac{e}{m^*} E_x t. \tag{4}$$

On average, an electron has traveled for a time τ since its last collision and thus

$$\langle v_x \rangle = -\frac{e}{m^*} E_x \tau. \tag{5}$$

The current density now simply follows as

$$\mathbf{j} = -en\langle v_x \rangle \hat{x} = \frac{e^2 n\tau}{m^*} E_x \hat{x}. \tag{6}$$

Of course, for an electric field along any general direction this can be generalized to

$$\mathbf{j} = \frac{e^2 n\tau}{m^*} \mathbf{E}. \tag{7}$$

3. **Landauer-Büttiker formalism and the quantum Hall effect. (16 points)**

   a. (4 points) With only a single Landau level below E_F in the bulk region of the 2DEG, the transport of electrons take place via a single edge state, to the right along the upper edge and to the left along the lower edge. The direct transmission sums are

   $$T_{21} = T_{32} = T_{43} = T_{14} = 1, \tag{8}$$

   and all the others are zero. Putting this into the Büttiker-Landauer equations yields

   $$I_1 = \frac{2e^2}{h}(V_1 - V_4), \tag{9}$$
   $$I_2 = \frac{2e^2}{h}(V_2 - V_1), \tag{10}$$
   $$I_3 = \frac{2e^2}{h}(V_3 - V_2), \tag{11}$$
   $$I_4 = \frac{2e^2}{h}(V_4 - V_3). \tag{12}$$

   With the given definition of the matrix γ, we have

   $$\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \tag{13}$$

b. (4 points) With terminals 1 and 3 as source and drain, respectively, the applied bias is $V_1 - V_3$, and we may, e.g., choose $V_3 = 0$. If terminals 2 and 4 are ideal voltage probes, we have immediately $I_2 = I_4 = 0$, which, by Kirchhoff's current rule implies $I_3 = -I_1$. The set of equations then reads

$$I_1 = \frac{2e^2}{h}(V_1 - V_4), \tag{14}$$

$$0 = \frac{2e^2}{h}(V_2 - V_1), \tag{15}$$

$$-I_1 = \frac{2e^2}{h}(V_3 - V_2), \tag{16}$$

$$0 = \frac{2e^2}{h}(V_4 - V_3). \tag{17}$$

and we have, from the second equation, $V_2 = V_1$ and, from the fourth equation, $V_4 = V_3 = 0$. Thus, the Hall resistance is

$$R_{\mathrm{H}} = \frac{V_2 - V_4}{I_1} = \frac{V_1}{I_1} = \frac{h}{2e^2}, \tag{18}$$

and the 2-terminal resistance is

$$R_{2t} = \frac{V_1 - V_3}{I_1} = \frac{V_1}{I_1} = \frac{h}{2e^2}. \tag{19}$$

c. (4 points) Interchange of the roles of terminals 2 and 3, i.e., with terminal 2 as drain and terminal 3 as voltage probe, we have $I_3 = 0$ and $I_2 = -I_1$. The equations are now

$$I_1 = \frac{2e^2}{h}(V_1 - V_4), \tag{20}$$

$$-I_1 = \frac{2e^2}{h}(V_2 - V_1), \tag{21}$$

$$0 = \frac{2e^2}{h}(V_3 - V_2), \tag{22}$$

$$0 = \frac{2e^2}{h}(V_4 - V_3). \tag{23}$$

Here, we choose $V_2 = 0$ and obtain $V_2 = V_3 = V_4 = 0$. The 2-terminal resistance is

$$R_{2t} = \frac{V_1 - V_2}{I_1} = \frac{V_1}{I_1} = \frac{h}{2e^2}, \tag{24}$$

as in (b). The longitudinal resistance is

$$R_L = \frac{V_3 - V_4}{I_1} = 0. \tag{25}$$

d. (4 points) When the magnetic field strength is reduced, we also reduce the separation $\hbar\omega_c = \hbar eB/m^*$ between the Landau levels in the bulk region of the 2DEG. Each time an additional Landau level falls below the Fermi energy E_F, a new edge state becomes available for electron transport, and the nonzero elements of the matrix γ increase by one (in absolute value). As a consequence, the Hall resistance R_H drops from one plateau to another, from the initial value $h/2e^2$ when we have one edge state at E_F, to $h/4e^2$, $h/6e^2$, and so on. The transition between subsequent Hall plateaus is more or less "smooth," depending on the probability of scattering from one side of the sample to the other when a bulk Landau level lines up with the Fermi energy.

The longitudinal resistance remains zero, or close to zero, for most values of B. However, an increased probability of backscattering when a bulk Landau level lines up with the Fermi energy yields a nonzero R_L, and we observe so called Shubnikov-de Haas oscillations.

4. **Fano factor of diffusive conductor. (5 points)**

a. (2 points) At low temperature, the conductance is given by

$$G = G_c \sum_n T_n = \int_0^1 dx\, x \sum_n \delta(x - T_n), \tag{26}$$

and we thus find

$$\langle G \rangle = G_c \int_0^1 dx\, \frac{\langle G \rangle}{2G_c} \frac{x}{x\sqrt{1-x}} = G_c \frac{\langle G \rangle}{2G_c} 2 = \langle G \rangle. \tag{27}$$

b. (2 points) For the expected shot-noise power we have similarly

$$\begin{aligned} \langle S_\mathrm{sn} \rangle &= 2eG_cV \left\langle \sum_n T_n(1 - T_n) \right\rangle \\ &= 2eG_cV \int_0^1 dx\, \frac{\langle G \rangle}{2G_c} \sqrt{1-x} \\ &= \frac{2}{3} eV \langle G \rangle = \frac{2}{3} e \langle I \rangle. \end{aligned} \tag{28}$$

c. (1 point) The expected Fano factor resulting from measuring the noise and the conductance follows as

$$\langle F \rangle = \frac{\langle S_\mathrm{sn} \rangle}{2e\langle I \rangle} = \frac{1}{3}. \tag{29}$$

5. **Weak antilocalization. (14 points)**

a. (4 points) For $\hat{R}_z(\phi)$ we find

$$\begin{aligned} e^{-\frac{i}{2}\phi\hat{\sigma}_z} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i\phi}{2}\right)^n \begin{pmatrix} 1^n & 0 \\ 0 & (-1)^n \end{pmatrix} \\ &= \sum_{n=0}^{\infty} \begin{pmatrix} \frac{1}{n!}\left(-\frac{i\phi}{2}\right)^n & 0 \\ 0 & \frac{1}{n!}\left(\frac{i\phi}{2}\right)^n \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}, \end{aligned} \tag{30}$$

and similarly we write for $\hat{R}_y(\phi)$

$$e^{-\frac{i}{2}\phi\hat{\sigma}_y} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\phi}{2}\right)^n \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^n. \tag{31}$$

We now note that for any integer n,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{2n} = (-1)^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{32}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{2n+1} = (-1)^n \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{33}$$

So we write

$$\begin{aligned} e^{-\frac{i}{2}\phi\hat{\sigma}_y} &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}\left(\frac{\phi}{2}\right)^{2n} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad + \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!}\left(\frac{\phi}{2}\right)^{2n+1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi/2 & -\sin\phi/2 \\ \sin\phi/2 & \cos\phi/2 \end{pmatrix}. \end{aligned} \tag{34}$$

b. (4 points)

$$\begin{aligned} \hat{U}_r &= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{-i(\alpha+\beta)/2}\cos\theta/2 & -e^{-i(\alpha-\beta)/2}\sin\theta/2 \\ e^{i(\alpha-\beta)/2}\sin\theta/2 & e^{i(\alpha+\beta)/2}\cos\theta/2 \end{pmatrix}. \end{aligned} \tag{35}$$

c. (2 points) We have

$$|s'\rangle = \hat{U}_r\,|s\rangle\,, \tag{36}$$

and

$$|s''\rangle = \hat{U}_r^{-1}|s\rangle = \hat{U}_r^\dagger|s\rangle\,, \tag{37}$$

where we used that $\hat{U}_r$ is unitary. This finally gives

$$\langle s''|s'\rangle = \langle s|\hat{U}_r^2|s\rangle. \tag{38}$$

Note that the inverse rotation $\hat{U}_r^{-1}$ is not equal to $\hat{U}_r$ with just replacing $\alpha \to -\alpha$, $\beta \to -\beta$, and $\theta \to -\theta$ because the rotation operators do not commute. Expressing everything explicitly in terms of the rotation operators, you could write

$$\hat{U}_r = \hat{R}_z(\alpha)\hat{R}_y(\theta)\hat{R}_z(\beta), \tag{39}$$
$$\hat{U}_r^{-1} = \hat{R}_z(-\beta)\hat{R}_y(-\theta)\hat{R}_z(-\alpha) = \hat{U}_r^\dagger. \tag{40}$$

d. (2 points) We find

$$\hat{U}_r^2 = \begin{pmatrix} e^{-i(\alpha+\beta)}\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} & -\frac{1}{2}(e^{-i\alpha}+e^{i\beta})\sin\theta \\ \frac{1}{2}(e^{i\alpha}+e^{-i\beta})\sin\theta & e^{i(\alpha+\beta)}\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \end{pmatrix}. \tag{41}$$

This yields explicitly

$$\begin{aligned}\langle s''|s'\rangle = {} & (|a|^2e^{-i(\alpha+\beta)} + |b|^2e^{i(\alpha+\beta)})\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \\ & -\frac{1}{2}\sin\theta\big[a^*b(e^{-i\alpha}+e^{i\beta}) - ab^*(e^{i\alpha}+e^{-i\beta})\big]. \end{aligned}\tag{42}$$

We see that averaging over all α, β, and θ makes all terms vanish except the last one on the first line, which gives $-\frac{1}{2}$.

e. (2 points) On average, the interference of phase-coherent time-reversed loops gives thus rise to a negative contribution to the probability of backscattering, which is therefore reduced. The conductivity is thus slightly higher than what one would expect from a classical (or Boltzmann) calculation. In the absence of spin-orbit interaction, one actually finds that the interference of time-reversed paths along closed loops, such as shown in the figure, is constructive. In that case, the probability of backscattering is thus slightly enhanced compared to the classical result. In total, under the assumptions made in the problem, the conductivity is thus always higher in spin-orbit coupled materials than in materials without spin-orbit coupling.

In total 55 points.