Department of physics, NTNU
TFY4340/FY8909 Nanophysics

Exam with solutions, June 7th, 2018

1. **Various qualitative questions. (6 points)**

   Use only a few sentences to answer each question.

   a. (2 points) Explain what distinguishes a metal, an insulator, and a semiconductor, in terms of the electronic band structure.

   SOLUTION:

   Metal: The Fermi energy E_{F} lies in the middle of a band (or at a level where multiple bands overlap), leading to a large density of states at E_{F}. When applying a small bias voltage, there are many states available for electrons which want to flow from source to drain, resulting in a large conductivity.

   Insulator: The Fermi energy lies deep inside a band gap, where the density of states is zero. A small bias voltage will thus not give rise to any current.

   Semiconductor: The Fermi energy lies inside a band gap, but this gap is not too large ($\sim$ 1-3 eV). It is easy to excite electrons into the (otherwise empty) conduction band, either by thermal activation or by doping. These excited electrons can give rise to a finite conductivity.

   b. (2 points) Explain what a Schottky barrier is.

   SOLUTION:

   From Wikipedia: "A Schottky barrier, named after Walter H. Schottky, is a potential energy barrier for electrons formed at a metal-semiconductor junction. Schottky barriers have rectifying characteristics, suitable for use as a diode. One of the primary characteristics of a Schottky barrier is the Schottky barrier height, denoted by Φ_B. The value of Φ_B depends on the combination of metal and semiconductor. Not all metal-semiconductor junctions form a rectifying Schottky barrier; a metal-semiconductor junction that conducts current in both directions without rectification, perhaps due to its Schottky barrier being too low, is called an Ohmic contact."

   In the language of the lecture notes (see p. 35): A Schottky barrier is an effective potential barrier created by a bending of the band structure close to the metal-semiconductor interface due to charge transfers across the interface that occur in order to "align" the work function of the metal and the electron affinity in the semiconductor to the same vacuum level.

c. (2 points) Describe the Aharonov-Bohm effect and explain qualitatively how it arises.
SOLUTION:

If a charged particle can travel, say, from A to B along two different paths, and there is a finite magnetic field penetrating the "loop" enclosed by these two paths, then there will be quantum interference in the probability $P_{A\to B}$ to travel from A to B depending on the strength of the magnetic field. This manifests itself as field-dependent oscillations in $P_{A\to B}$, which is what we call the Aharonov-Bohn effect.

Indeed, a finite field corresponds to a position-dependent vector potential $\mathbf{A}(\mathbf{r})$, which means that the wave function of the particle picks up a phase $(-e/\hbar)\int \mathbf{A}\cdot d\mathbf{l}$ that can be different along the two paths and depends on the magnitude of the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. In fact, one finds that the interference effect depends only on the flux $\Phi = \int \mathbf{B}\cdot d\mathbf{S}$ penetrating the loop (where $\mathbf{S}$ is the area of the loop), and the resulting oscillations in the probability are periodic in Φ with a period $\Phi_0 = h/e$, the flux quantum.

2. **Chemical potential for a two-dimensional electron gas. (4 points)**

a. (2 points) Calculate the density of states

$$D_2(E) = \frac{dN}{dE},$$

with $N(E)$ the number of allowed states with energy E or smaller, for a two-dimensional free electron gas contained in an area A.
SOLUTION:

The number of electronic states $N(k)$ with wave vector k or smaller is

$$N(k) = 2\frac{A}{4\pi^2}\pi k^2 = \frac{Ak^2}{2\pi},$$

where the first factor of 2 is for spin. In terms of energy $E(k) = \hbar^2 k^2/2m$ this yields

$$N(E) = \frac{mA}{\pi\hbar^2}E,$$

and the density of states thus reads

$$D_2(E) = \frac{mA}{\pi\hbar^2}.$$

b. (2 points) Express the chemical potential μ at temperature T in terms of the total electron density n.
Hint:

$$\int_0^\infty dx\,\frac{1}{ae^x+1} = \ln\left(1+\tfrac{1}{a}\right)$$

SOLUTION:
The total electron density at temperature T reads

$$n = \frac{1}{A}\int_0^\infty D_2(E)f(E),$$

with

$$f(E) = \frac{1}{e^{(E-\mu)/k_\mathrm{B}T}+1}$$

being the Fermi function. Using the hint we find explicitly

$$n = k_\mathrm{B}T\frac{m}{\pi\hbar^2}\ln\left(1+e^{\mu/k_\mathrm{B}T}\right),$$

which allows to solve for μ,

$$\mu = k_\mathrm{B}T\ln\left[\exp\left(\frac{\pi n\hbar^2}{mk_\mathrm{B}T}\right)-1\right].$$

3. **Statistics of rare electron transfers. (6 points)**

We consider the statistics of electron transfers through a tunnel junction where all transmission probabilities $T_n \ll 1$ are small. This means that we can assume all successful electron transfers through the junction to be uncorrelated, which allows us to find a simple expression for the characteristic function describing the transfers.

We start by considering a very short time interval dt, during which the chance for a transfer is $dt/\tau \ll 1$, where $1/\tau$ is the transfer rate.

a. (1 point) Show that the characteristic function of the probability distribution $P_{N,dt}$ for counting N transfers within time dt can be approximated

$$\Lambda_{dt}(\chi) = \langle e^{i\chi N}\rangle \approx \exp\left\{dt(e^{i\chi}-1)/\tau\right\}.$$

SOLUTION:
We have

$$\begin{aligned}\Lambda_{dt}(\chi) = \langle e^{i\chi N}\rangle &= \sum_N P_{N,dt}e^{i\chi N}\\ &\approx (1-dt/\tau)+(dt/\tau)e^{i\chi}\\ &= 1+dt(e^{i\chi}-1)/\tau\\ &\approx \exp\left\{dt(e^{i\chi}-1)/\tau\right\}.\end{aligned}$$

b. (1 point) Assuming all transfers to be uncorrelated, write down the characteristic function $\Lambda_{\Delta t}(\chi)$ for a larger time interval Δt in terms of the average number of transfers $\langle N \rangle = \Delta t/\tau$.

SOLUTION:

For uncorrelated events we simply have

$$\begin{aligned}\Lambda_{\Delta t}(\chi) &= \Lambda_{dt}(\chi)^{(\Delta t/dt)} = \exp\left\{\Delta t(e^{i\chi}-1)/\tau\right\} \\ &= \exp\left\{\langle N \rangle (e^{i\chi}-1)\right\}.\end{aligned}$$

c. (1 point) Show that the distribution function $P_{N,\Delta t}$ describing the probability to have N transfers occurring in the time interval Δt is a Poisson distribution,

$$P_{N,\Delta t} = e^{-\langle N \rangle}\frac{\langle N \rangle^N}{N!}.$$

SOLUTION:

$\Lambda_{\Delta t}(\chi)$ and $P_{N,\Delta t}$ are each other's Fourier transform. We thus evaluate

$$\begin{aligned}P_{N,\Delta t} &= \int_0^{2\pi}\frac{d\chi}{2\pi}\,\Lambda_{\Delta t}(\chi)e^{-i\chi N} \\ &= \int_0^{2\pi}\frac{d\chi}{2\pi}\,\exp\left\{\langle N \rangle (e^{i\chi}-1)\right\}e^{-i\chi N} \\ &= e^{-\langle N \rangle}\int_0^{2\pi}\frac{d\chi}{2\pi}\,e^{-i\chi N}e^{\langle N \rangle e^{i\chi}} \\ &= e^{-\langle N \rangle}\int_0^{2\pi}\frac{d\chi}{2\pi}e^{-i\chi N}\sum_{k=0}^{\infty}\frac{\langle N \rangle^k e^{ik\chi}}{k!} \\ &= e^{-\langle N \rangle}\sum_{k=0}^{\infty}\frac{\langle N \rangle^k}{k!}\int_0^{2\pi}\frac{d\chi}{2\pi}e^{i\chi(k-N)} \\ &= e^{-\langle N \rangle}\sum_{k=0}^{\infty}\frac{\langle N \rangle^k}{k!}\delta_{k,N} \\ &= e^{-\langle N \rangle}\frac{\langle N \rangle^N}{N!}.\end{aligned}$$

d. (1 point) Find the probability to have no transfers in a time interval Δt.

SOLUTION:

$$P_{0,\Delta t} = e^{-\langle N \rangle} = e^{-\Delta t/\tau}.$$

e. (1 point) Calculate the second cumulant $\langle\langle N^2 \rangle\rangle_{\Delta t}$ for the time interval Δt.

SOLUTION:

The n-th cumulant follows from the generating function as

$$\langle\langle N^n \rangle\rangle_{\Delta t} = \frac{\partial^n}{\partial (i\chi)^n} \ln \Lambda_{\Delta t}(\chi)\bigg|_{\chi\to 0}.$$

The moment-generating function $\Lambda_{\Delta t}(\chi)$ for the Poisson distribution reads

$$\ln \Lambda_{\Delta t}(\chi) = \langle N \rangle (e^{i\chi} - 1),$$

and we immediately see that all cumulants are equal,

$$\langle\langle N^2 \rangle\rangle_{\Delta t} = \langle\langle N^3 \rangle\rangle_{\Delta t} = \langle\langle N^n \rangle\rangle_{\Delta t} = \langle N \rangle.$$

f. (1 point) Find the Fano factor

$$F = \frac{\langle\langle N^2 \rangle\rangle_{\Delta t}}{\langle\langle N \rangle\rangle_{\Delta t}},$$

for a tunnel junction in this Poisson limit.
SOLUTION:
From (e) we see that $F = 1$.

4. **Joule heating in the Drude model. (6 points)**

The Drude model uses the assumption that all collisions between electrons and impurities randomize the direction of motion of the electron, $\langle v_{x,y,z} \rangle = 0$ directly after a collision. Another assumption, which we did not discuss in the lectures, is that the collisions are *inelastic*: It is assumed that each collision reduces the kinetic energy of the electron to the same value $\frac{1}{2}mv_i^2$ (which is related to the temperature of the sample). Indeed, if we would have assumed elastic collisions, the kinetic energy of the electrons would constantly increase in the presence of an electric field, growing almost indefinitely for large samples.

a. (1 point) We assume an electric field E_x present, which points in the x-direction. As we know, the effect of this field is to shift v_x by an amount $-eE_x t/m$ during a time t. Calculate the gain in kinetic energy during time t

$$\Delta E(t) = \frac{1}{2} m \big[v(t)^2 - v(0)^2 \big],$$

if the initial velocity is $\mathbf{v}(0) = v_i(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Show that assuming the *direction* of $\mathbf{v}(0)$ to be random yields an average energy gain during time t

$$\langle \Delta E(t) \rangle = \frac{e^2 E_x^2 t^2}{2m}.$$

SOLUTION:

With an initial velocity $\mathbf{v}(0) = v_i(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ we find

$$\mathbf{v}(t) = v_i(\sin\theta\cos\phi - eE_x t/m, \sin\theta\sin\phi, \cos\theta).$$

We then see that

$$v(t)^2 - v(0)^2 = -2\frac{eE_x t}{m}\sin\theta\cos\phi + \frac{e^2E_x^2t^2}{m^2},$$

and thus

$$\Delta E(t) = -eE_x t\sin\theta\cos\phi + \frac{e^2E_x^2t^2}{2m}.$$

Assuming the initial direction to be random, we see that the first term averages out,

$$\langle\Delta E(t)\rangle = \frac{e^2E_x^2t^2}{2m}.$$

b. (1 point) The Drude model assumes all collisions with impurities to be uncorrelated. This means that the statistics of these collisions are exactly the same as those of the electron transfers through a tunnel junction considered in problem 3, and we can thus use the results we obtained there: If the probability for an electron to collide with an impurity in an infinitesimal time interval dt is dt/τ, where $1/\tau$ is the collision rate, then the probability that a randomly picked electron has suffered no collisions during the preceding time t is $e^{-t/\tau}$.

Argue why the probability density function for the time between two successive collisions for a single electron reads

$$p(t) = \frac{1}{\tau}e^{-t/\tau}.$$

Hint: In terms of actual probabilities, $p(t)dt$ is the probability to find a time between two collisions in the interval $[t, t+dt]$.

SOLUTION:

(i) The probability for an electron to collide with an impurity within the tiny time interval dt is dt/τ. (ii) The probability for an electron to travel without collisions for a time t is $e^{-t/\tau}$. Therefore, the probability that an electron travels freely for a time t and then collides in the tiny time interval $[t, t+dt]$ is the product of these two probabilities: $(dt/\tau)e^{-t/\tau}$. The corresponding probability density function for the times t between two collisions is thus

$$p(t) = \frac{1}{\tau}e^{-t/\tau}.$$

c. (1 point) Calculate the average time between two collisions.

SOLUTION:

This can simply be done with the probability density function found above:

$$\langle t\rangle = \int_0^\infty dt\, t\, p(t) = \frac{1}{\tau}\int_0^\infty dt\, te^{-t/\tau} = \tau$$

d. (1 point) Use the results from (a) and (b) to calculate the average energy an electron gains between two collisions.
SOLUTION:
The energy gained as a function of the time between two collisions is $e^2 E_x^2 t^2/2m$, so the expectation value for this energy reads

$$\frac{e^2 E_x^2}{2m}\langle t^2 \rangle = \frac{e^2 E_x^2}{2m\tau}\int_0^\infty dt\, t^2 e^{-t/\tau} = \frac{e^2 E_x^2 \tau^2}{m}$$

e. (1 point) Assuming that all this energy gained between two collisions is transfered to the lattice at the second collision, show that the average energy transfer from the electrons to the sample per volume per second reads

$$\sigma E_x^2,$$

where $\sigma = ne^2\tau/m$ is the conductivity, with n being the electron density.
SOLUTION:
The average collision frequency per volume in the sample is n/τ, where n is the electron density, and on average each collision transfers an energy $e^2 E_x^2 \tau^2/m$ from the electrons to the lattice. The total rate of energy transfer per volume is thus

$$\frac{ne^2\tau}{m} E_x^2 = \sigma E_x^2,$$

where $\sigma = ne^2\tau/m$ is the conductivity

f. (1 point) Show that this yields for the total energy dissipated per second in a conductive sample of length L and cross section A the familiar result

$$P = I^2 R,$$

where $R = L/\sigma A$ is the total resistance of the sample.
SOLUTION:
The total energy dissipated per second is thus

$$P = AL\sigma E_x^2,$$

where AL (cross section times length) is the volume of the sample. The current density (along the direction of the field) is

$$j = \sigma E_x,$$

and the total current reads therefore

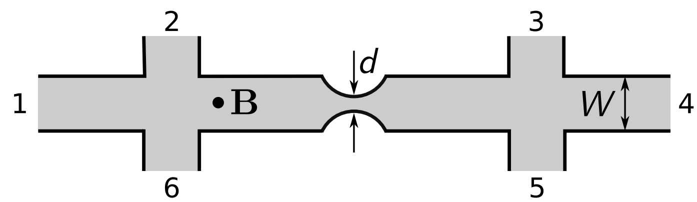
$$I = A\sigma E_x.$$

Inserting $E_x = I/A\sigma$ and using that $R = L/\sigma A$ one easily finds

$$P = I^2 R.$$

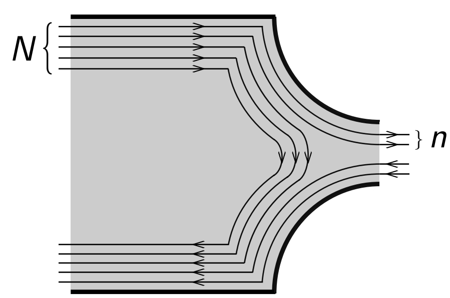
5. **Landauer-Büttiker formalism and the quantum Hall effect. (6 points)**

Consider the following 6-terminal Hall bar:



The bar has a width W, but in the middle there is a constriction of width d. A uniform magnetic field $\mathbf{B}$ is applied perpendicularly to the 2DEG and points out of the plane, leading to N edge states in the wide regions of the device. Inside the constriction there are only $n < N$ edge states available at the Fermi level. We thus assume that there are n channels with perfect transmission in each direction through the constriction; the other $N - n$ edge states in the left and right parts of the device are not connected.

The following zoom-in might help to visualize the configuration of the edge channels near the constriction, where I picked $N = 5$ and $n = 2$:



The Landauer-Büttiker equations,

$$I_\alpha = \sum_{\beta \neq \alpha} G_{\alpha\beta}(V_\alpha - V_\beta),$$

with conductances

$$G_{\alpha\beta} = \frac{2e^2}{h} T_{\alpha\beta},$$

relate the net current entering into terminal α to the potentials at the various terminals. Here, $T_{\alpha\beta}$ denotes the "direct transmission sum" from terminal β to terminal α.

a. (2 points) Neglecting impurity scattering and tunneling processes, list all non-zero $T_{\alpha\beta}$ and give their magnitudes.
SOLUTION:
We have

$$T_{21} = T_{16} = T_{43} = T_{54} = N, \\ T_{62} = T_{35} = N - n, \\ T_{32} = T_{65} = n.$$

b. (4 points) Let terminals 1 and 4 be the "source" and the "drain" respectively through which a current I runs, whereas the other terminals are ideal voltage probes. Find

- the Hall resistance $R_{14,35} = (V_3 - V_5)/I$;
- the Hall resistance $R_{14,26} = (V_2 - V_6)/I$;
- the Hall resistance $R_{14,25} = (V_2 - V_5)/I$;
- the Hall resistance $R_{14,36} = (V_3 - V_6)/I$;
- the longitudinal resistance $R_{14,23} = (V_2 - V_3)/I$;
- the longitudinal resistance $R_{14,65} = (V_6 - V_5)/I$;
- the 2-terminal resistance $R_{14,14} = (V_1 - V_4)/I$.

For convenience you can set $V_4 = 0$.
Advice: This looks like a lot of contacts to consider and resistances to calculate. But don't panic, just write down the correct equations for all the I_α and you will see that everything is rather simple to solve.
SOLUTION:
We start by writing down the Büttiker-Landauer equations (where for convenience of notation we introduce $i = hI/2e^2$):

$$\begin{aligned} i &= N(V_1 - V_6), \\ 0 &= N(V_2 - V_1), \\ 0 &= N(V_3 - V_5) + n(V_5 - V_2), \\ -i &= -NV_3, \\ 0 &= NV_5, \\ 0 &= N(V_6 - V_2) + n(V_2 - V_5), \end{aligned}$$

where we already have set $V_4 \to 0$. We see: (i) $V_5 = 0$ and (ii) $V_2 = V_1$, leading to

$$\begin{aligned} i &= N(V_1 - V_6), \\ 0 &= NV_3 - nV_1, \\ -i &= -NV_3, \\ 0 &= N(V_6 - V_1) + nV_1, \end{aligned}$$

so we find

$$
\begin{aligned}
V_1 = V_2 &= \frac{hI}{2e^2}\frac{1}{n}, \\
V_3 &= \frac{hI}{2e^2}\frac{1}{N}, \\
V_4 = V_5 &= 0, \\
V_6 &= \frac{hI}{2e^2}\left(\frac{1}{n} - \frac{1}{N}\right).
\end{aligned}
$$

This yields straightforwardly

$$
\begin{aligned}
R_{14,35} &= \frac{V_3 - V_5}{I} = \frac{h}{2Ne^2}, \\
R_{14,26} &= \frac{V_2 - V_6}{I} = \frac{h}{2Ne^2}, \\
R_{14,25} &= \frac{V_2 - V_5}{I} = \frac{h}{2ne^2}, \\
R_{14,36} &= \frac{V_3 - V_6}{I} = \frac{h}{2e^2}\left(\frac{2}{N} - \frac{1}{n}\right), \\
R_{14,23} &= \frac{V_2 - V_3}{I} = \frac{h}{2e^2}\left(\frac{1}{n} - \frac{1}{N}\right), \\
R_{14,65} &= \frac{V_6 - V_5}{I} = \frac{h}{2e^2}\left(\frac{1}{n} - \frac{1}{N}\right), \\
R_{14,14} &= \frac{V_1 - V_4}{I} = \frac{h}{2ne^2}.
\end{aligned}
$$